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Influence of slip velocity on the flow of viscous fluid through a porous medium in a permeable tube with a variable bulk flow rate

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ABSTRACT

A mathematical model is presented for an axisymmetric flow of a Newtonian fluid through a permeable tube filled with porous medium and slip at the wall. The bulk flow rate is prescribed as a decreasing function of axial distance. The governing coupled partial differential equations are solved analytically using Adomian decomposition method and numerically using a second order finite difference scheme. Numerical method is validated by already published work and a good agreement is observed between the two solutions. Trusting this validity, effects of pertinent parameters on the flow variables such as velocity components, wall shear stress and pressure drop are discussed graphically. This study reveals that the slip parameter (ϕ), reabsorption parameter (α) and permeability parameter (k) have significant influences on flow variables involved in the problem. Creeping flow ($Re = 0$), flow without porous medium ($k \to \infty$) and flow with no slip ($\phi = 0$) at the wall are the limiting cases of this study.

Introduction

The study on the flow in permeable ducts is of practical interest in biological and engineering problems [1–5]. This is an idealization of the behavior of flow that occurs in everyday life in corresponding geometries. Many processes such as transpiration cooling, membrane filtration, gaseous diffusion in binary mixtures, physiological blood flow in artery and vein, flow in proximal tubule of a kidney and artificial dialysis may be modeled by the flow in permeable ducts.

Theoretical and experimental models representing filtration processes are proposed by many investigators. Berman [6] for the first time, investigated the behavior of steady laminar flow of an incompressible fluid through a permeable channel. He obtained expressions for velocity profile and pressure drop through second order perturbation by assuming uniform wall suction. Yuan et al. [7,8] extended the work of Berman [6] for small and large seepage rates. Terrill [9] studied the two-dimensional flow problem in a porous pipe and found an exact solution. Jocelyne et al. [10] carried out a detailed study on the laminar flow in porous channels and discussed variations in pressure drop, permeation flow, axial and transversal velocity profiles along the length of the filter channel. He considered rectangular channel with one permeable wall and a permeable tubular channel to study the non-symmetrical transverse flow as well as symmetrical radial flow. Further, Sandeep [11] examined the flow in both a rectangular slit and a tube with porous boundaries and solved the problem analytically to obtain pressure drop of fluid.

The processes of ultrafiltration and reverse osmosis have applications in various parts of human body and their uses in extracorporeal processing of fluids in the body. In these processes the fluid is actually pumped through permeable ducts at elevated pressure. Many researchers investigated the flow through permeable channels/ tubes with application to flow in human kidney. Kelman [12] carried out theoretical study on the flow in nephrons by considering normal velocity as an exponentially decaying function of downstream distance. Macey [13,14] obtained solutions for creeping flows with the possibility of linear and exponential variations of absorption rates with axial distance. However his solutions were incomplete. Later, Kosoniski [15] provided complete solutions and extended the work for flow between parallel slits. Muthu and Berhane [16,17] investigated flow in channel with absorbing walls and non-uniform cross-section. They applied perturbation method to study the mathematical model related to flow in human kidney. Marshal and Trowbridge [18] used physical

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conditions instead of prescribing the permeation velocity at the tube wall and calculated leakage flux and fractional reabsorption. Radhakrishnamacharya et al. [19] examined the flow problem of Newtonian fluid in a porous tube of varying cross-section with an application to flow in renal tubule of human kidney.

In the preceding studies, researchers assumed the axial component of velocity to be equal to zero at the permeable boundaries. Although this is a widely used condition for the analysis of flows past solid boundaries but it has no empirical justification in the case of flow over a permeable boundaries. It has been now accepted by many investigators that most of the polymeric materials tends to slip on the solid walls. A detailed study on the flow of Johnson-Segalman fluid is done by Rao and Rajagopal [20] which provides information about the effects of slip condition on the Newtonian fluid flow. Moustafa [21] discussed the importance of considering slip at the boundaries. This is known from literature that slip velocity and shear rate are directly proportional at the boundary. Slip velocity is actually connected to a thin layer of the streamwise flowing fluid just beneath the permeable boundary. Fluid in the thin layer is pulled along by the fluid present above the porous boundary. Moreover, consideration of slip is also useful in describing certain problems in many other applications [22–28].

Furthermore, in above mentioned studies no attention has been given to the flow through porous medium. However it is known from literature review that many vascular and biological tissues, the blood vessels and renal system are assumed to be porous by nature. Khaled [29] provided a detailed review on fluid flow through porous medium having physiological applications. Therefore it is concluded that effects of porous medium together with slip effects on the flow characteristics seem to be important and thus deserve special attention.

Fluid models consisting of permeable ducts with constant flux in normal direction are not an appropriate choice for the analysis of flow problems which do not involve uniform flow in the normal direction at the boundaries. Recently many authors [30–32] investigated the physiological flows using different geometries with the assumption of variable flow rate in axial direction resulting from variable flux at the boundaries. They found the solutions both analytically and numerically. Very recently the researchers [33,34] studied the blood flow problems in circular tubes and obtained analytical solution using a technique namely, Adomian decomposition method (ADM) [35]. This technique provides computable and accurate solutions for sufficiently small number of terms. One of the advantages of this method is that it does not require simplifications which some times lead to change the physical behavior of the flow models. This method tackles the problems in a simple and straightforward way without linearization, perturbation or any restrictive assumption which results in physically more realistic solutions [35–39]. Moreover, the second order finite difference method is applied and considered as the numerical solution for validity of ADM.

The increasing number of biophysical and industrial flow problems forces us to extend the available hydrodynamic solutions in order to include all possible issues and find solutions using applicable analytical techniques. Keeping in mind the above discussion, purpose of this work is to analyze the two-dimensional flow of a Newtonian fluid in the presence of porous medium in a tube with slip at the wall. Flow equations are considered with non-zero Reynolds number and flow rate being a decreasing function of axial distance. Investigation of influence of Reynolds number Re, permeability parameter k and slip coefficient ϕ on various flow parameters is the major concern of this work. This study provides an extended form of already available results in literature so that published results can be achieved by appropriate substitutions of flow parameters. Present work is helpful in providing information for the improvement of available models for the solution of different biophysical and engineering flow problems.

This manuscript is arranged as follows: Section 1 provides introduction of the problem. In Section 2, hydrodynamical equations and non-uniform boundary conditions are presented. In Section 3, an approximate solution of the problem is obtained. Section 4 is for results and discussion and concluding remarks are presented in Section 5.

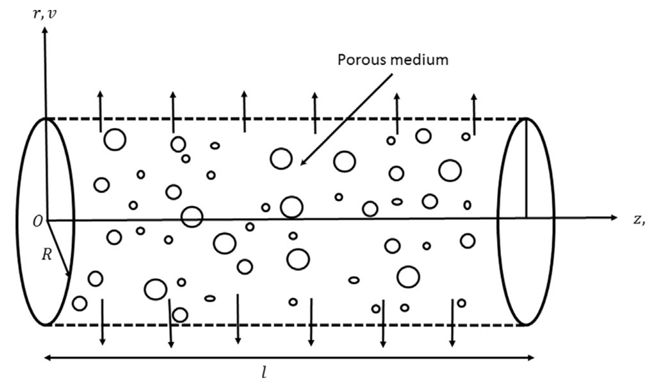

Fig. 1. Schematic diagram of flow through a porous medium in a tube.

Formulation of the problem

A steady laminar flow of an incompressible Newtonian fluid is considered in a two-dimensional permeable tube filled with porous medium as depicted in Fig. 1. The tube of radius R is considered long enough in order to neglect the entrance and end effects. A partial slip condition is assumed at the permeable wall of the tube and the bulk flow rate is considered as a decreasing function of axial distance. The continuity and momentum equations governing such flow are [30–34]:

$$\frac{\partial u'}{\partial z'} + \frac{1}{r'}\frac{\partial (r'v')}{\partial r'} = 0, \tag{1}$$

$$\rho\left(u'\frac{\partial u'}{\partial z'} + v'\frac{\partial u'}{\partial r'}\right) = -\frac{\partial p'}{\partial z'} + \mu\left[\frac{\partial^2 u'}{\partial z'^2} + \frac{1}{r'}\frac{\partial}{\partial r'}\left(r'\frac{\partial u'}{\partial r'}\right) - \frac{u'}{k'}\right], \tag{2}$$

$$\rho\left(u'\frac{\partial v'}{\partial z'} + v'\frac{\partial v'}{\partial r'}\right) = -\frac{\partial p'}{\partial r'} + \mu\left[\frac{\partial^2 v'}{\partial z'^2} + \frac{\partial}{\partial r'}\left(\frac{1}{r'}\frac{\partial}{\partial r'}(r'v')\right) - \frac{v'}{k'}\right], \tag{3}$$

where $u'(z', r')$ and $v'(z', r')$ are velocity components in axial z' and radial r' directions respectively, ρ is the density, μ is the viscosity, $p'(z', r')$ is the pressure of the fluid and k' is the permeability of porous medium. Appropriate boundary conditions for the problem under consideration are

Regularity condition [30–34]:

$$\frac{\partial u'}{\partial r'} = 0, \quad v' = 0, \quad \text{at} \quad r' = 0. \tag{4}$$

Slip at the boundary [28]:

$$u' + \phi'\left(\frac{\partial u'}{\partial r'} + \frac{\partial v'}{\partial z'}\right) = 0, \quad \text{at} \quad r' = R. \tag{5}$$

The reabsorption at the wall is prescribed by considering bulk flow as a decreasing function of downstream distance. That is the flow rate across a cross-section of the tube is given by [30–32]

$$Q'(z') = 2\pi\int_0^R r'u'\left(z', r'\right)dr' = Q_0F\left(\alpha'z'\right), \tag{6}$$

where $F(\alpha'z') = 1$ for $\alpha' = 0$ and decreases with z';$\alpha' \geqslant 0$ is the reabsorption coefficient; Q_0 is the flux across the cross-section at $z' = 0$.

Pressure at the inlet of the tube is

$$p'(x', y') = p_0' \quad \text{at} \quad z' = 0, r' = 0. \tag{7}$$

Boundary condition (5) is the slip-flow condition proposed by Beavers and Joseph [23]. In Eq. (5), $\phi' = \frac{\sqrt{k_0'}}{\beta}$ is the slip coefficient where k_0' is the constant wall permeability and β is the dimensionless constant depending on the wall characteristics.

Introducing the following dimensionless quantities

$$z = \frac{z'}{l}, \quad r = \frac{r'}{R}, \quad u = \frac{2\pi R^2}{Q_0}u', \quad v = \frac{2\pi Rl}{Q_0}v', \quad k = \frac{k'}{R^2}, \quad \alpha = \alpha' l \quad k_0 = \frac{k_0'}{R^2}. \tag{8}$$

Eqs. (1)–(3) reduce to

$$\frac{\partial u}{\partial z} + \frac{1}{r}\frac{\partial (rv)}{\partial r} = 0, \tag{9}$$

$$Re\delta\left(u\frac{\partial u}{\partial z} + v\frac{\partial u}{\partial r}\right) = -\frac{\partial p}{\partial z} + \delta^2\frac{\partial^2 u}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) - \frac{u}{k}, \tag{10}$$

$$Re\delta^3\left(u\frac{\partial v}{\partial z} + v\frac{\partial v}{\partial r}\right) = -\frac{\partial p}{\partial r} + \delta^2\left[\delta^2\frac{\partial^2 v}{\partial z^2} + \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv)\right) - \frac{v}{k}\right], \tag{11}$$

where $Re = \frac{Q_0}{2\pi R\nu}$ is the Reynolds number, $p = \frac{2\pi R^4}{\mu Q_0 l}p'$ is the dimensionless pressure and $\delta = \frac{R}{l}$ is the ratio of radius to the length of the tube.

Defining vorticity function as

$$\Omega = \frac{\partial u}{\partial r} - \delta^2\frac{\partial v}{\partial z}. \tag{12}$$

Eqs. (10)–(11) in terms of vorticity function are

$$Re\delta\left[v\Omega + \frac{1}{2}\frac{\partial}{\partial z}(u^2 + v^2)\right] = -\frac{\partial p}{\partial z} + \left(\frac{\partial}{\partial r} + \frac{1}{r}\right)\Omega - \frac{u}{k}, \tag{13}$$

$$Re\delta^3\left[-u\Omega + \frac{1}{2}\frac{\partial}{\partial z}(u^2 + v^2)\right] = -\frac{\partial p}{\partial r} - \frac{\partial \Omega}{\partial z} - \frac{v}{k}, \tag{14}$$

We further define stream function as

$$u\left(z, r\right) = -\frac{1}{r}\frac{\partial \psi(z, r)}{\partial r}, \quad v\left(z, r\right) = \frac{1}{r}\frac{\partial \psi(z, r)}{\partial z}. \tag{15}$$

Using Eq. (15) into Eqs. (13)–(14) and then eliminating p between the resulting equations, we arrive at the following single partial differential equation

$$E^4\psi = rRe\delta\left[\frac{\partial \psi}{\partial z}\frac{\partial}{\partial r}\left(\frac{E^2\psi}{r^2}\right) - \frac{\partial \psi}{\partial r}\frac{\partial}{\partial z}\left(\frac{E^2\psi}{r^2}\right)\right] + \frac{1}{k}E^2\psi, \tag{16}$$

where

$$E^2 = \delta^2\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r}.$$

Boundary conditions in terms of stream function are as follows

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial r}\right) = 0, \quad \psi = 0, \quad \text{at} \quad r = 0,$$

$$\frac{1}{r}\frac{\partial \psi}{\partial r} = \phi\left[-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial r}\right) + \frac{1}{r}\frac{\partial^2 \psi}{\partial z^2}\right], \quad \psi = -F\left(\alpha z\right), \quad \text{at} \quad r = 1, \tag{17}$$

where $\phi = \phi'/R$.

We assume $F(\alpha z) = e^{-\alpha z}$. This assumption is important from physiological point of view suggested in [12] and followed by many investigators [16,17,19,30–32].

Solution of the problem

We solve the problem (16)–(17) analytically by Adomian decomposition method (ADM) [35–39] and numerically by a second order finite difference scheme. A comparison is made between the two solutions in the form of a Table 2 which shows a good agreement.

Analytical solution

For analytical solution, we define a linear operator as

$$L = \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r}, \tag{18}$$

Then Eq. (16) can be rewritten as

$$L^2\psi = Re\delta N\psi - \delta^4\frac{\partial^4 \psi}{\partial z^4} - 2\delta^2 L\frac{\partial^2 \psi}{\partial z^2} + \frac{1}{k}\left(L\psi + \delta^2\frac{\partial^2 \psi}{\partial z^2}\right), \tag{19}$$

where

$$N\psi = r\left[\frac{\partial \psi}{\partial z}\frac{\partial}{\partial r}\left(\frac{E^2\psi}{r^2}\right) - \frac{\partial \psi}{\partial r}\frac{\partial}{\partial z}\left(\frac{E^2\psi}{r^2}\right)\right], \tag{20}$$

is the non-linear part.

Now, operating L^{-2} on both sides of Eq. (19), we obtain

$$\psi = \psi_0 + L^{-2}\left[Re\delta N\psi - \delta^4\frac{\partial^4 \psi}{\partial z^4} - 2\delta^2 L\frac{\partial^2 \psi}{\partial z^2} + \frac{1}{k}\left(L\psi + \delta^2\frac{\partial^2 \psi}{\partial z^2}\right)\right]. \tag{21}$$

To proceed further, we first find inverse operator L^{-2}. For this, consider equation $L\psi = G$. This equation has a solution

$$\psi = [L_1^{-1}r(L_1^{-1}r^{-1})]G, \tag{22}$$

where L_1^{-1} is one fold integral such that $L_1^{-1} = \int(\cdot)dr$.

Eq. (22) shows that

$$L_1^{-1} = [L_1^{-1}r(L_1^{-1}r^{-1})], \tag{23}$$

which further gives

$$L^{-2} = L_1^{-1}[rL_1^{-1}(r^{-1}L_1^{-1}r(rL_1^{-1}r^{-1}))]. \tag{24}$$

In Eq. (21), ψ_0 represents the solution of homogeneous equation $L\psi_0 = 0$ which is given as

$$\psi_0\left(x, y\right) = \frac{r^4}{4}a(z) + \left(\frac{\ln r}{2} - \frac{1}{4}\right)r^2 b(z) + \frac{r^2}{2}c(z) + d(z), \tag{25}$$

In above equation $a(z)$, $b(z)$, $c(z)$, $d(z)$ are constants which are to be determined from the given boundary conditions. We now, decompose u and $N\psi$ as follows [36]

$$\psi = \sum_{n=0}^{\infty}\psi_n, \quad N\psi = \sum_{n=0}^{\infty}A_n, \tag{26}$$

where A_n are the Adomian polynomials. Further, decomposing ψ_0 as follows

$$\psi_0 = \sum_{n=0}^{\infty}\psi_{0,n}, \tag{27}$$

where we have applied double decomposition [36]. Using Eqs. (24)–(27) into Eq. (21), we arrive at

$$\psi_{n+1} = \psi_{0,n+1} + L^{-2}\left[Re\delta A_n - \delta^4\frac{\partial^4 \psi_n}{\partial z^4} - 2\delta^2 L\frac{\partial^2 \psi_n}{\partial z^2} + \frac{1}{k}\left(L\psi_n + \delta^2\frac{\partial^2 \psi_n}{\partial z^2}\right)\right]. \tag{28}$$

From Eqs. (25) and (27), we may write

$$\psi_{0,n+1} = \frac{r^4}{4}a_n(z) + \left(\frac{\ln r}{2} - \frac{1}{4}\right)r^2 b_n(z) + \frac{r^2}{2}c_n(z) + d_n(z), \tag{29}$$

where the constants have also been decomposed as

$$a(z) = \sum_{n=0}^{\infty}a_n(z), \quad b(z) = \sum_{n=0}^{\infty}b_n(z),$$
$$c(z) = \sum_{n=0}^{\infty}c_n(z), \quad d(z) = \sum_{n=0}^{\infty}d_n(z). \tag{30}$$

From Eq. (28), we have

$$\psi_1 = \psi_{0,1} + L^{-2}\left[Re\delta A_0 - \delta^4\frac{\partial^4 \psi_0}{\partial z^4} - 2\delta^2 L\frac{\partial^2 \psi_0}{\partial z^2} + \frac{1}{k}\left(F\psi_0 + \delta^2\frac{\partial^2 \psi_0}{\partial z^2}\right)\right], \tag{31}$$

where

$$\psi_{0,1} = \frac{r^4}{4}a_1(z) + \left(\frac{\ln r}{2} - \frac{1}{4}\right)r^2 b_1(z) + \frac{r^2}{2}c_1(z) + d_1(z). \quad (32)$$

The Adomian polynomials $A_0, A_1, A_2, ...A_n$ are generated in such way [36] that

$$A_0 \equiv A_0(\psi_0), \quad A_1 \equiv A_1(\psi_0, \psi_1),$$
$$A_2 \equiv A_2(\psi_0, \psi_1, \psi_2), \ ... \quad A_n \equiv A_n(\psi_0, \psi_1, \psi_2, ..., \psi_n).$$

Substituting Eq. (26) into Eq. (20), we arrive at

$$A_n = r\left[\frac{\partial \psi_n}{\partial z}\frac{\partial}{\partial r}\left(\frac{E^2\psi_n}{r^2}\right) - \frac{\partial \psi_n}{\partial r}\frac{\partial}{\partial z}\left(\frac{E^2\psi_n}{r^2}\right)\right], \quad (33)$$

from where

$$A_0 = r\left[\frac{\partial \psi_0}{\partial z}\frac{\partial}{\partial r}\left(\frac{E^2\psi_0}{r^2}\right) - \frac{\partial \psi_0}{\partial r}\frac{\partial}{\partial z}\left(\frac{E^2\psi_0}{r^2}\right)\right]. \quad (34)$$

Boundary conditions take the following form

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi_0}{\partial r}\right) = 0, \quad \psi_0 = 0, \quad \text{at} \quad r = 0,$$
$$\frac{1}{r}\frac{\partial \psi_0}{\partial r} = \phi\left[-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi_0}{\partial r}\right) + \frac{1}{r}\frac{\partial^2 \psi_0}{\partial z^2}\right], \quad \psi_0 = -e^{-\alpha z}, \quad \text{at} \quad r = 1, \quad (35)$$

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi_1}{\partial r}\right) = 0, \quad \psi_1 = 0, \quad \text{at} \quad r = 0,$$
$$\frac{1}{r}\frac{\partial \psi_1}{\partial r} = \phi\left[-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi_1}{\partial r}\right) + \frac{1}{r}\frac{\partial^2 \psi_1}{\partial z^2}\right], \quad \psi_1 = 0, \quad \text{at} \quad r = 1. \quad (36)$$

Solving Eq. (25) along with the boundary conditions (35), we get the final form of ψ_0 as

$$\psi_0 = f(z)r^4 + g(z)r^2, \quad (37)$$

where

$$f(z) = \frac{(2 - \phi\delta^2\alpha^2)e^{-\alpha z}}{2(4\phi + 1)},$$
$$g(z) = -\frac{[4 + (8 - \delta^2\alpha^2)\phi]e^{-\alpha z}}{2(4\phi + 1)}. \quad (38)$$

From Eqs. (31)–(32) ψ_1 is determined as

$$\psi_1 = \frac{r^4}{4}a_1(z) + \left(\frac{\ln r}{2} - \frac{1}{4}\right)r^2 b_1(z) + \frac{r^2}{2}c_1(z) + d_1(z) + \Omega(z)r^{10} + \xi(z)r^8 + \eta(z)r^6, \quad (39)$$

where

$$\Omega(z) = \frac{Re\delta^3(f_1 f_2 - 2ff_3)}{2520}, \quad (40)$$

$$\xi(z) = -\frac{\delta}{1680k}[\delta^3 kf_4 + 2kRe\delta^2(f_3 g + 2fg_3 - f_2 g_1) - \delta f_2 + 32kReff_1], \quad (41)$$

$$\eta(z) = -\frac{1}{360k}[2kRe\delta g(8f_1 + \delta^2 g_3) + k\delta^2(\delta^2 g_4 + 16f_2) - \delta^2 g_2 - 8f]. \quad (42)$$

In Eqs. (41)–(42), subscripts with f and g represent the order of derivative with respect to z. $a_1(z)$, $c_1(z)$ are obtained from boundary conditions (36) as

$$a_1(z) = -\frac{4}{1 + 4\phi}[4(1 + 10\phi)\Omega + 3(1 + 8\phi)\xi + 2(1 + 6\phi)\eta], \quad (43)$$

$$c_1(z) = \frac{2}{1 + 4\phi}[3(1 + 12\phi)\Omega + 2(1 + 10\phi)\xi + (1 + 8\phi)\eta]. \quad (44)$$

Thus two term approximate solution is

$$\psi = \psi_0 + \psi_1$$
$$= \Omega r^{10} + \xi r^8 + \eta r^6 + mr^4 + nr^2, \quad (45)$$

where

$$m(z) = f(z) + \frac{a_1(z)}{4},$$
$$n(z) = g(z) + \frac{c_1(z)}{2}. \quad (46)$$

Now velocity components can be obtained from Eqs. (12) and (42) as

$$u = 10\Omega r^8 + 8\xi r^6 + 6\eta r^4 + 4mr^2 + 2n,$$
$$v = \Omega_1 r^9 + \xi_1 r^7 + \eta_1 r^5 + m_1 r^3 + n_1 r. \quad (47)$$

Here superscripts with Ω, ξ, η, m and n represent the first derivative with respect to z.

We see that expression for streamfunction is similar to the expressions already available in literature [33,34].

Pressure distribution

Pressure distribution for the problem under consideration can be found by integrating Eqs. (10) and (11) as

$$p\left(z, r\right) = \delta^2\frac{\partial u}{\partial z} + \int\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) - \frac{u}{k}\right]dz - Re\delta\left[\int u\frac{\partial u}{\partial z}dz + \int v\frac{\partial u}{\partial r}dz\right] + \text{a constant of integration}. \quad (48)$$

Mean pressure drop is defined as

$$\bar{p}(z) = \frac{\int_0^1 2\pi r p(z, r)dr}{\pi}. \quad (49)$$

Therefore, mean pressure drop between $z = 0$ and $z = z_0$ can be written as

$$\Delta\bar{p}(z_0) = \bar{p}(0) - \bar{p}(z_0), \quad (50)$$

Wall shear stress

Wall shear stress τ_w in dimensionless form is defined as

$$\tau_w = -\left(\frac{\partial u}{\partial r} + \frac{\partial v}{\partial z}\right)_{r=1}. \quad (51)$$

Upon use of Eq. (47), we get

$$\tau_w = -8[10\Omega + 6\xi + 3\eta + m]. \quad (52)$$

Numerical solution

To get a numerical solution by finite difference scheme of second order, we rewrite Eq. (16) as follows

$$\delta^2\frac{\partial^2\zeta}{\partial z^2} - \frac{1}{r}\frac{\partial\zeta}{\partial r} + \frac{\partial^2\zeta}{\partial r^2} = -\delta Re\left[\frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial\zeta}{\partial z} + \frac{2}{r^2}\frac{\partial\psi}{\partial z}\zeta - \frac{1}{r}\frac{\partial\psi}{\partial z}\frac{\partial\zeta}{\partial r}\right] + \frac{1}{k}\zeta, \quad (53)$$

where ζ is given by

$$\zeta = -r\Omega = \delta^2\frac{\partial^2\psi}{\partial z^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial r^2}. \quad (54)$$

The solution procedure described here is based on Eqs. (53)–(54) involving the stream function ψ and the vorticity function ζ as variables. The boundary conditions are given in Eq. (17). An iterative finite difference solution procedure is discussed here. A series of evenly spaced nodal lines are used as shown in Fig. 2. The grid spacings are thus constant in both the $z-$ and $r-$directions and are equal to Δz and Δr respectively.

Consider the nodal points shown in Fig. 3. In terms of the values at these nodal points the finite-difference form of Eqs. (53) and (54) are

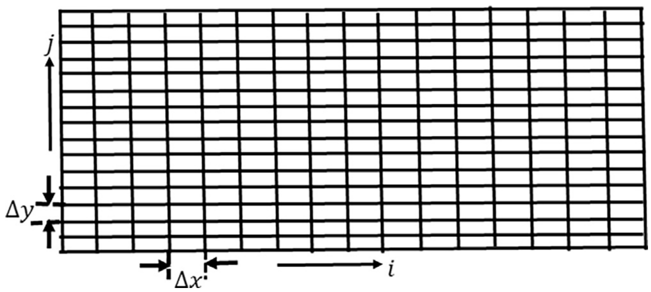

Fig. 2. Grid system used.

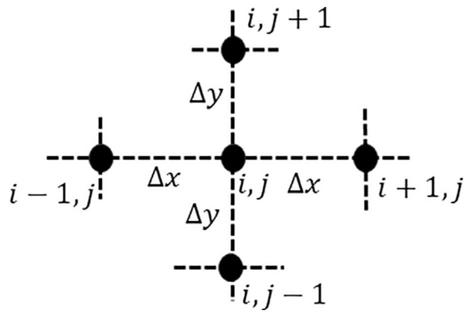

Fig. 3. Nodal points used in obtaining finite difference equations.

$$\delta^2\left(\frac{\zeta_{i+1,j}-2\zeta_{i,j}+\zeta_{i-1,j}}{\Delta z^2}\right)-\frac{1}{r_{i,j}}\left(\frac{\zeta_{i,j+1}-\zeta_{i,j-1}}{2\Delta r}\right)+\left(\frac{\zeta_{i,j+1}-2\zeta_{i,j}+\zeta_{i,j-1}}{\Delta r^2}\right)$$
$$=-\delta Re\begin{bmatrix}\frac{1}{r_{i,j}}\left(\frac{\psi_{i,j+1}-\psi_{i,j-1}}{2\Delta r}\right)\left(\frac{\zeta_{i+1,j}-\zeta_{i-1,j}}{2\Delta z}\right)\\+\frac{2}{r_{i,j}^2}\left(\frac{\psi_{i+1,j}-\psi_{i-1,j}}{2\Delta z}\right)\zeta_{i,j}\\-\frac{1}{r_{i,j}}\left(\frac{\psi_{i+1,j}-\psi_{i-1,j}}{2\Delta z}\right)\left(\frac{\zeta_{i,j+1}-\zeta_{i,j-1}}{2\Delta r}\right)\end{bmatrix}+\frac{1}{k}\zeta_{i,j} \quad (55)$$

$$\zeta_{i,j}=\delta^2\left(\frac{\psi_{i+1,j}-2\psi_{i,j}+\psi_{i-1,j}}{\Delta z^2}\right)-\frac{1}{r_{i,j}}\left(\frac{\psi_{i,j+1}-\psi_{i,j-1}}{2\Delta r}\right)+\left(\frac{\psi_{i,j+1}-2\psi_{i,j}+\psi_{i,j-1}}{\Delta r^2}\right), \quad (56)$$

The following boundary condition given in (17)

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)=0$$

is satisfied if $\psi_{i,2}= \psi_{i,1}$. Since $\psi = 0$ and $r = 0$, so that $\psi_{i,2} = \psi_{i,1} = 0$ and from Eq. (54), we have

$$-\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{\partial^2\psi}{\partial r^2}=\zeta,$$

from where we find that $\zeta_{i,1} = 0$, $i = 1, 2, ..., M$. At $r = 1$, $\psi = F(\alpha z) = e^{-\alpha z}$, so that Eq. (54) gives us

$$\delta^2\frac{\partial^2\psi}{\partial z^2}=\zeta,$$

which leads to $\zeta_{i,N} = \phi\delta^2\alpha^2 e^{-\alpha z i}$.

The set of dimensionless nonlinear finite difference equations (55)–(56) combined with the associated boundary conditions are solved iteratively starting from the guessed values of the variables at all points. As r varies from 0 to 1, ψ varies from 0 to $e^{=\alpha z}$, therefore we assume that the values of ψ varies linearly so that we may have guessed values of ψ at the internal nodal points

$$\psi_{i,j}=\frac{j=1}{N-1}=e^{=\alpha z}, \quad i = 1, 2, ..., M \text{ and } j = 2, 3, ..., N = 1.$$

and using the Eq. (54), we find the guessed values of $\zeta_{i,j}$.

To find new values of $\zeta_{i,j}$, Eq. (53) is applied sequentially at all internal nodal points. Right-hand side of Eq. (53) is found by using the assumed values of variables. Further, under-relaxation is used so that "updated" values of $\zeta_{i,j}$ are actually taken as

$$\zeta_{i,j}^1=\zeta_{i,j}^0+r(\zeta_{i,j}^{calculated}-\zeta_{i,j}^0),$$

where r is the relaxation factor (< 1) and the value $\zeta_{i,j}^{calculated}$ is given directly by Eq. (53). Here the subscripts 0 and 1 represent the conditions at the beginning and end of the iteration step respectively. This step is repeated before we proceed to the next step of the procedure. This process actually accelerates the convergence.

In the next step, similar procedure is done with the Eq. (54). Again under-relaxation is used so that the "updated" values of $\psi_{i,j}$ are actually taken as

$$\psi_{i,j}^1=\psi_{i,j}^0+r(\psi_{i,j}^{calculated}-\psi_{i,j}^0),$$

where r is the relaxation factor (< 1) and the value $\psi_{i,j}^{calculated}$ is given directly by Eq. (54). This step is also repeated before we proceed to the next step of the procedure.

Above mentioned steps are repeated again and again until the values of variables cease to change from one iteration to the next by less than some prescribed value. Once the distributions of ψ and ζ are found, the distributions of u, v and p are actually been found.

Results and discussions

The present analysis is carried out in order to understand the behavior of steady two-dimensional laminar flow of an incompressible Newtonian fluid. A permeable tube filled with porous medium is taken for the flow analysis by considering bulk flow rate as an exponentially decaying function of downstream distance. Flow variables such as axial and radial components of velocity u and v respectively, wall shear stress (τ_w) and mean pressure drop (Δp) are computed numerically and influences of various parameters such as reabsorption parameter (α), permeability parameter (k), and slip coefficient (ϕ) are discussed through graphs. A comparison of this work with the already published work [32] shows a very good match (Table 1). Recall that the effects of porous medium and reabsorption at the wall in the presence of slip are the prime objectives of this study. In this analysis, we have chosen the moderate values of pertinent parameters as $\delta = 0.1$, $\alpha = 1.5$, $Re = 1$, $\phi = 0.25$ and $k = 0.1$.

The effects of flow parameters on axial velocity component (u) and radial velocity component (v) are shown graphically. Effects of α, k and ϕ on u are shown in Figs. 4–6 at a cross section $z = 0.3$ of the tube. It can be observed from Fig. 4 that u is a decreasing function of α. Physically this means that u decreases because of more reabsorption from the wall. It is observed from Fig. 5 that u increases with the permeability parameter k up to $r = 0.6$ and a reverse trend is seen for $r > 0.6$. Fig. 6 reveals that ϕ produces significant influence on axial velocity. It is clear that axial velocity decreases near the center whereas it increases near the wall by increasing ϕ. Figs. 7–9 show the effects of α, k and ϕ on radial velocity. It is observed that radial velocity increases significantly

Table 1
Comparison of present study and published work for pressure drop over the length of the tube for limiting case of $\phi = 0$ and $k \to \infty$ when $Re = 1$, $\delta = 0.1$ and $z = 1$.

α	Published work [32]	ADM	Numerical
1	9.61315	9.61315	9.61315
1.1	9.18640	9.18640	9.18639
1.2	8.78600	8.78600	8.78599
1.3	8.40990	8.40990	8.40989
1.4	8.05623	8.05622	8.05623

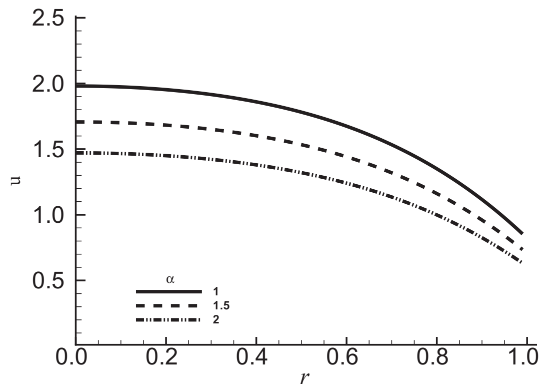

Fig. 4. Effect of α on axial velocity.

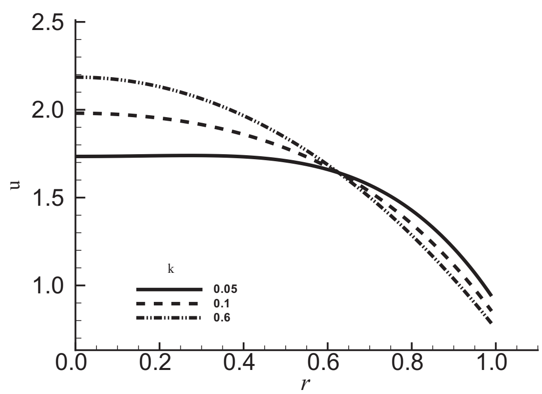

Fig. 5. Effect of k on axial velocity.

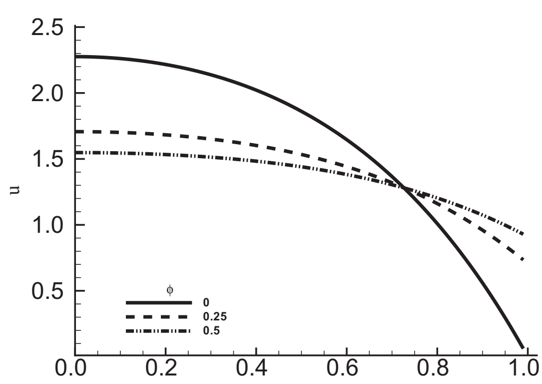

Fig. 6. Effect of ϕ on axial velocity.

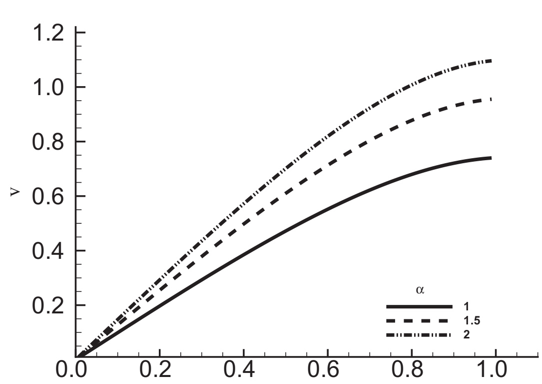

Fig. 7. Effect of α on radial velocity.

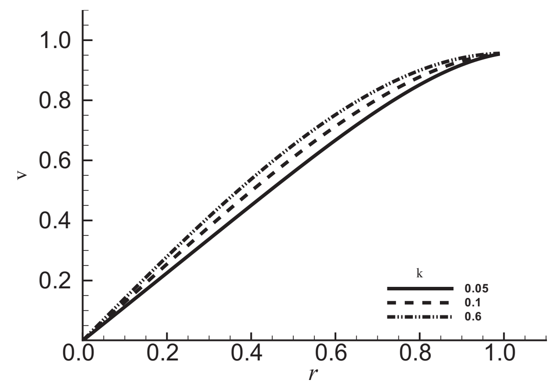

Fig. 8. Effect of k on radial velocity.

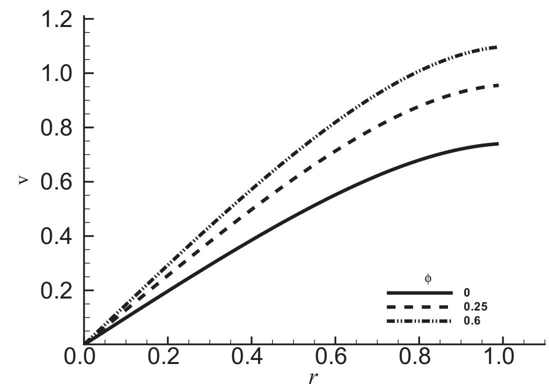

Fig. 9. Effect of ϕ on radial velocity.

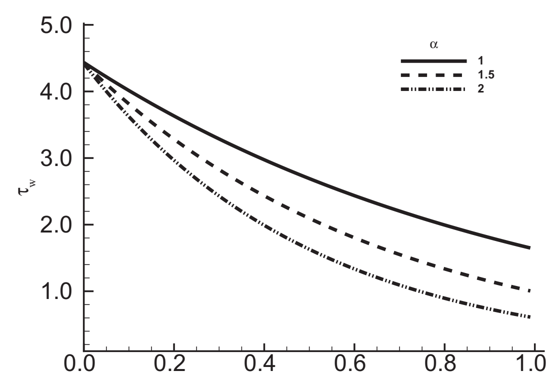

Fig. 10. Effect of α on shear stress.

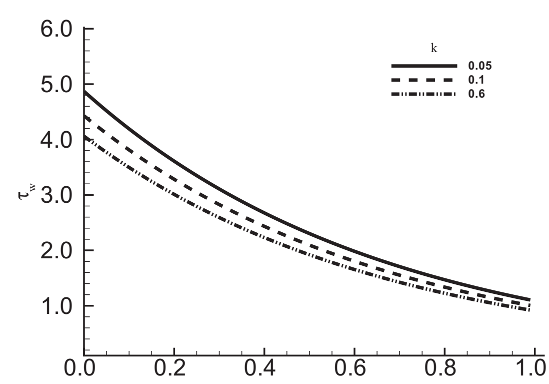

Fig. 11. Effect of k on shear stress.

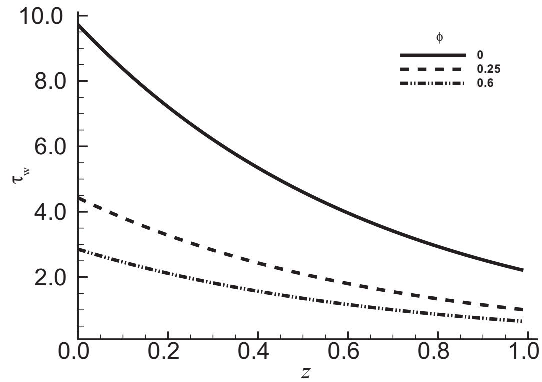

Fig. 12. Effect of ϕ on shear stress.

with the increase of α and k (Figs. 7–8). Furthermore, radial velocity decreases with the increase of ϕ (Fig. 9).

Effects of α, k and ϕ on wall shear stress τ_w are represented in Figs. 10–12. Wall Shear stress decreases in all the cases. By increasing α, decrease in wall shear stress is more significant near the wall than near the center. Further, wall shear stress significantly decreases by increasing k and ϕ.

Moreover, mean pressure drop is calculated over the whole length of the tube for various values of α, k and ϕ. As shown in Figs. 13–15, mean pressure drop decreases by increasing all the parameters α, k and ϕ. Permeability parameter k have a great influence on mean pressure drop. It tends to be negative for higher values of k which shows that reverse flow is expected for high values of k.

A comparison between the published [32] and present work can be seen from Table 1 where a good agreement can be seen for the values of mean pressure drop. Moreover, the parameter α shows the same behavior on the flow as observed in previous studies [30–32].

Table 2 shows a comparison between analytical and numerical solutions. A good comparison for different values of slip parameter ϕ can be seen from this table.

Conclusions

In this study, problem of an incompressible Newtonian fluid flow through a porous medium in a permeable tube with partial slip at the wall is investigated. The novel features of this study are the consideration of porous medium inside the permeable tube together with slip at the wall particularly to the renal flow. Moreover full Navier Stokes equations are attempted to solve analytically and numerically.

Following are the major findings concerned with the present study.

- Axial velocity increases with increase in k up to a certain distance from the centre of the channel. However it decreases as we move towards the wall of the channel. Also u decreases when we increase the values of α and ϕ.
- Radial velocity increases with increasing α and k while it decreases by increasing ϕ.
- Shear stress and mean pressure drop decrease with increasing α, ϕ and k.
- Solutions, disregarding inertial effects can be achieved as limiting case by choosing $Re = 0$.
- The published work [32] is a special case of the present work when no slip condition ($\phi = 0$) and no porous medium ($k \to \infty$) is considered.
- Numerical results are validated with ADM and published work [32].
- A second order finite difference method is used for the validity of ADM and a good agreement has been observed by comparing ADM solution with the numerical solution.
- The present study may readily be extended for the flows in tubes with varying cross sections.

The present study possibly bears the potential to investigate the change that takes place in velocity, shear stress and pressure drop of the fluid due to α, k and ϕ. It is interesting to note that α, k and ϕ have significant influences on velocity, shear stress and pressure drop.

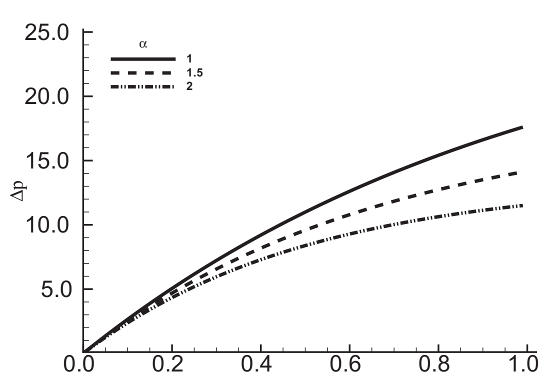

Fig. 13. Effect of α on pressure drop.

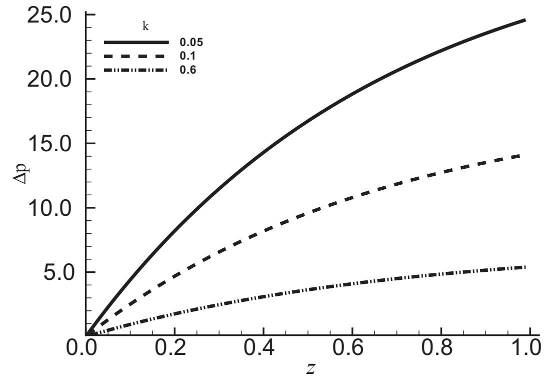

Fig. 14. Effect of k on pressure drop.

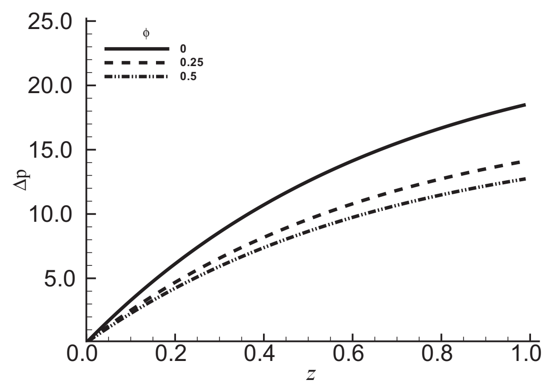

Fig. 15. Effect of ϕ on pressure drop.

Table 2
Comparison of ADM and numerical method for pressure drop when $Re = 1$, $\delta = 0.1$, $\alpha = 1.5$, $k = 0.1$ and $z = 1$.

ϕ	ADM	Numerical	Percentage error
0	7.72301	7.72332	3.88×10^{-2}
0.1	5.48652	5.48656	7.29×10^{-3}
0.2	4.23362	4.23363	2.36×10^{-3}
0.3	3.43231	3.43237	1.75×10^{-2}
0.4	2.87577	2.87581	1.39×10^{-2}

This study provides information for any flow situations occurring in engineering or biophysical problems where fluid flows through porous medium and bulk flow rate decreases with axial distance. Hopefully model presented here is an improvement of already available models and may be useful in filtration processes or in the treatment of certain disorders of the kidney and cardiovascular systems.

Conflicts of interest

The authors declare that they have no conflict of interest.

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Appendix A. Supplementary data

Supplementary data associated with this article can be found, in the online version, at https://doi.org/10.1016/j.rinp.2018.10.049.

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