Analysis of heat and mass transfer for Casson fluid along a moving sheet in the presence of thermal radiation and suction/injection effects

Thesis Submitted By

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A dissertation submitted to the Department of Computer Science, Bahria University, Islamabad as a partial fulfillment of the requirements for the award of the degree of MS in Mathematics

Session (2017 - 2019)

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Dedicated to

My respected parents and teachers

whose prayers and support have always been a source of encouragement for me

My caring and supporting wife and lovely children

have always given me care and love.

Acknowledgments

All praise be to **ALLAH** the Almighty, the Merciful and the Beneficent Who has enabled me to achieve milestones towards my destination and to complete my thesis. Peace be upon the Prophet Hazrat **Muhammad** Whose life has always been a source of an inspiration and guidance to all of us.

First of all, I am very much thankful to my supervisor **Dr. Rizwan ul Haq** due to whom I have been able to accomplish my thesis. He is a great and passionate person and has always been available to support and help in whatever problems be. I am placing my earnest thanks to Dr. Rizwan ul Haq. It is indeed an honour for me to work with such a great person.

I express my gratitude to all my honorable teachers whose guidance and supervision have brought me to this stage of academic zenith, in particular, Prof. Dr. Muhammad Ramzan and Dr. Jafar Hasnain who have always been supportive in all of my course work and kept encouraging me throughout the session in Bahria University, Islamabad Campus. They are the true teachers who have made Mathematics Department of BUIC, a real place of learning.

I express my deepest sense of gratitude to my family who are always the real pillars for my encouragement and showered their everlasting love, care and support throughout my life. I am very much thankful to my friends Ali Raza, Syed Saqib Shah, Syed Hammad Shah, Muhammad Sajid and Imran Abbas for their cooperation in my MS studies. Special thanks to some of my colleagues Prof. Naveed Akmal, Shams ul Haq (Lec.) and Abid Khan (Computer operator) who helped me throughout my MS.

Umar Ali Shah

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May 2019

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Nomenclature

V	velocity vector
ρ	density of fluid
$\bar{T}$	fluid's temperature
$\bar{\tau}$	Cauchy stress tensor
$\bar{C}$	Concentration of species
μ	dynamic viscosity
ρC_p	heat capacity
k	thermal conductivity
k_1	reaction rate coefficient
B_o	uniform magnetic field
k_2	rate of specific internal heat generation
$\bar{T}_\infty$	temperature at infinity
σ	electric conductivity
$\bar{C}_\infty$	concentration at infinity
α_1	reaction paramter
Nu	local Nusselt number
α_2	internal heat parameter
q_w	heat transfer
σ^*	thermal conductivity
Re	Reynolds number
D	mass diffusion coefficient
γ	Casson fluid parameter

Abstract

The emphasis of this thesis is to present a thorough concept of analysis of heat and mass transfer for Casson fluid along moving sheet in the presence of thermal radiation and suction/injection effects. Firstly, mathematical formulation is developed for momentum, energy and concentration profiles. Later on, these PDEs are transformed into the dimensionless non-linear ODEs by using similarity transformations and the boundary conditions are defined with the thermal radiation term for the energy equation which is described with two cases (PST and PHF). The acquired solution of the dimensionless nonlinear ODEs will be evaluated by shooting method numerically using MATLAB. Results thus obtained for different physical parameters, namely, magnetic and Casson fluid parameter, stagnation point, Prandtl number, internal heat generation parameter, Schmidt number and thermal radiation on each profile is being constructed and discussed in detail.

Chapter 1

Introduction and literature review

Researches are confined around the boundaries of surface that is authentic for very tiny viscosity or extreme Reynold's number. Andersson [13] established the closed form solution of complete N-S equations for magnetohydrodynamics flow in stretched surface by applying the similarity equations without the approximations of the thin layer on the boundary. Due to the dissipation of the term $\frac{\partial^2 u}{\partial x^2}$ from the N-S equation vertically, the results are same as of the results obtained by Pavlov [4], calculated in the structure of extreme Reynold's number. Beyond the horizontal momentum equation is opted to evaluate the pressure distribution that enhances with the normal distance from the surface. Since Andersson [13] has never discussed implication of the layers on the boundaries, that's why, the solution obtained by him is of closed-form and also authentic for every Reynold's number. Therefore, Wang [7] has gathered this solution as an exact solution for N-S equations.

Non-Newtonian is the nonlinear relationship between shear stress and deformation rate. For past decades, few substances in industries have significantly been used, such as foams, emulsions and suspensions are considered to be polymeric melts and solutions which are not assumed to be Newtonian postulates. These fluids are accordingly known as non-Newtonian. Some of the utilization of these fluids are briefly stated in following table by Chhabra [18].

Table 1: **Exhibition of non-Newtonian fluid behavior by some substances**

Adhesives(wall paper, paste, carpet)	Mine tailings and mineral suspensions
Biological fluid(blood, etc.)	Paints, Polish and varnish
Animal waste slurries from cattle farm	Greases and lubricating oils

The process is also applied in the Engineering and in accordance with the Probstein [11], "homogeneous or heterogeneous reactions usually lead to an important heat release occurred with non-isothermal conditions that need a suitable heat source term to be added in the heat transfer equation." Andersson et al. [12] examined that stretched surface may contain some dispersal of the chemically responsive species having isothermal condition and deduced that the first order reaction has the valid solution as the closed-form. However, the application of heat and mass transfer is brought at all the surfaces (Solid surface), it could better be studied when the nature of these effects are fallen out for homogeneous type or it is only being occurred at the huge fluid flow in the heat and mass transfer.

For non-isothermal stretched surface, the temperature distribution is being evaluated for two cases, "prescribed surface temperature (PST) and prescribed surface heat flux (PHF)," in which distance far from the origin has linear proportionality to the surface thermal conditions Beyond this, the problem of mass transfer is studied. Abramowitz and Stegun [2] have suggested its analytic solutions in Kummer's function with same conditions already being formulated in Andersson et al. [12]. In PST case, solution of temperature can be established to that of the concentration by switching some of the major parameters like α_1, Sc, α_2 and Pr.

Chapter 2

Elementary concepts of fluid

This chapter deals with the basic definitions related to the concepts of fluid are described in detail.

2.1 Fluid

A substance satisfying the deformation property under the application of stress is called fluid.

2.2 Fluid mechanics

A type of mechanics where fluid's flow is being studied. It can also be described whether the fluid is in motion or at rest and can be further classified into three types, they are; Fluid dynamics, kinematics and statics.

2.3 Fluid properties

There are some physical properties of fluid which are defined below:

2.3.1 Density

It is the change in the mass of fluid w.r.t its volume which is denoted by ρ and is presented as

$$\rho = \frac{m}{V}$$

2.3.2 Viscosity

The change in the shear stress w.r.t rate of the deformation of fluid is called viscosity. Greek letter $\bar{\mu}$ is symbol of viscosity. Mathematically

$$\bar{\mu} = \frac{shear\ stress}{deformation\ rate}$$

2.3.3 Kinematic viscosity

It is change in viscosity of fluid w.r.t. its density and it is denoted by ν. Mathematically it can be expressed as

$$\nu = \frac{\mu}{\rho}$$

2.4 Classification of fluid

Fluid is classified in the following two types.

2.4.1 Ideal fluid

Fluid with zero viscosity at each point is called ideal or inviscid fluid.

2.4.2 Real fluid

Fluid with non-zero viscosity at any point is known as real. Real fluid is consisting of two types.

2.4.2.1 Newtonian fluid

Newtonian fluid that obeys Newton's law of viscosity. Some of the examples of such type are Air, water and mercury. Mathematically;

$$\tilde{\tau}_{yx} = \mu \frac{d\bar{u}}{dy}$$

Here $\tilde{\tau}_{yx}$, $\bar{u}$, $\frac{d\bar{u}}{dy}$ and μ are the shear stress, velocity component, deformation rate and dynamic viscosity, respectively.

2.4.2.2 Non-Newtonian fluid

Non-Newtonian fluid fails to obey Newton's law of viscosity. Blood, paints and ketchup are some of its examples. Mathematically

$$\tilde{\tau}_{yx} = \bar{k}(\frac{d\bar{u}}{dy})^m$$

for $m \neq 1$, Here $\bar{k}$ and m are consistency and behavior index, respectively.

2.4.3 Compressible fluid

Fluid in which the density does not remain constant and changes with T and $\hat{p}$ like gases.

2.4.4 Incompressible fluid

Fluid in which density remains unchanged with T and $\hat{p}$ is called incompressible fluid. In general, all kind of liquids contain such properties of incompressible.

2.5 Hydromagnetic flow

Hydromagnetics deal with behavior of electrically conducting fluids and magnetic properties. Some of its examples are the plasmas, salt water and electrolytes, etc.

2.6 Prandtl number

Non-dimensional number which is change in kinematic viscosity w.r.t. thermal diffusivity. Mathematically,

$$Pr = \frac{\nu}{\alpha}$$

2.7 Reynolds number

It is dimensionless number which is a change in inertial forces to viscous forces. Mathematically

$$Re = \frac{ax^2}{\nu}$$

2.8 Schmidt number

A non-dimensional number defined as, "The ratio of kinematic viscosity and mass diffusivity is called Schmidt number." Mathematically

$$Sc = \frac{\nu}{D}$$

2.9 Thermal radiation

It is an electromagnetic radiation emitted from a material due to heat and its characteristics depends upon temperature, called thermal radiation.

2.10 Heat and mass transfer

It is transfer of heat from one place to another through the movement of particles of fluid which is basically a kinematic process. Similar motion of particles through mass is called mass transfer.

Chapter 3

Analysis of heat and mass transfer of hydromagnetic flow over a stretching surface

This chapter is the analysis of heat transfer of hydromagnetic flow passing via stretched surface with transverse magnetic field. Governing PDEs of momentum, energy and concentration are transformed into nonlinear ODEs by applying compatible similarity variables. Furthermore, the solution of nonlinear ODEs are computed by power series and attain results with discussion. This chapter is basically the review of [16].

3.1 Mathematical formulation and solution

Considering an electrically conducting incompressible fluid passing through a non-conducting stretched sheet $y = 0$ having a velocity bx for positive b and x as a horizontal coordinate. The fluid flows at $y > 0$ under the consideration of uniform B_o. On the fixed origin, the coupled forces are applied to stretch the sheet horizontally elsewhere the quiescent fluid's motion is just driven by stretching surface. Steady two dimensional equations of momentum, energy and concentration for hydromagnetic fluid [15, 19] will be established as

$$\bar{u}_x + \bar{v}_y = 0 \tag{3.1}$$

$$\rho(\bar{u}\bar{u}_x + \bar{v}\bar{u}_y) = -\hat{p}_x + \mu(\bar{u}_{xx} + \bar{u}_{yy}) - \sigma B_o^2 \bar{u} \tag{3.2}$$

$$\rho(\bar{u}\bar{v}_x + \bar{v}\bar{v}_y) = -\hat{p}_y + \mu(\bar{v}_{xx} + \bar{v}_{yy}) \tag{3.3}$$

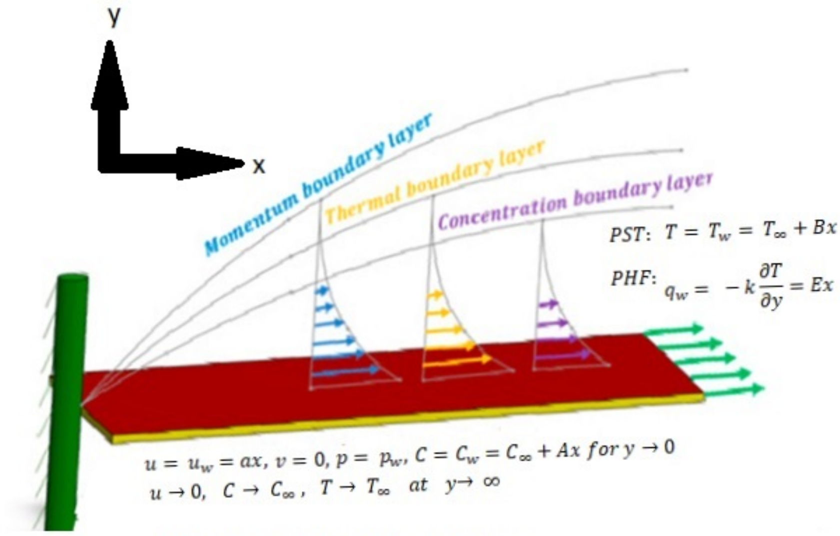


Fig. 1: Geometry of problem

$$\bar{u}\bar{C}_x + \bar{v}\bar{C}_y = D(\bar{C}_{xx} + \bar{C}_{yy}) + k_1(\bar{C} - \bar{C}_\infty) \tag{3.4}$$

$$\rho c_p(\bar{u}\bar{T}_x + \bar{v}\bar{T}_y) = k(\bar{T}_{xx} + \bar{T}_{yy}) + k_2(\bar{T} - \bar{T}_\infty) \tag{3.5}$$

By neglecting electric field, induced magnetic field gives smaller value as compare to applied magnetic field which results $-\sigma B_o^2\bar{u}$ from the assumption. In Eqs. (3.4) and (3.5), the positive values of k_1 and k_2 represent constructive chemical reaction of species and internal heat generation, respectively whereas the negative values of k_1 and k_2 are the destructive chemical reaction and internal heat absorption, respectively. It is clearly seen that temperature depends upon k_1 while k_1 is assumed to be constant by [16]. Temperature and concentration fields do not have a cross effect from the assumptions in Eqs. (3.4) and (3.5) by neglecting Dufour-Soret effects.

Boundary conditions are defined as

$$\bar{u} = u_w = bx, \quad \bar{v} = 0, \quad \hat{p} = \hat{p}_w, \quad \bar{C} = \bar{C}_w = \bar{C}_\infty + Bx,$$

$$\text{PST:} \quad \bar{T} = \bar{T}_w = \bar{T}_\infty + Ex,$$
$$\text{PHF:} \quad q_w = -k\bar{T}_y|_{y=0} = Ax \tag{3.6}$$
$$\bar{u} \to 0, \ \bar{C} \to \bar{C}_\infty, \ \bar{T} \to \bar{T}_\infty \ \text{ at } \ y \to \infty \tag{3.7}$$

The above two equations depict B.Cs for the concentration and energy equations.

3.2 Methods of Solution

3.2.1 Momentum transfer problem

Transformation, to calculate the exact solution of given hydromagnetic boundary value problem, can be introduced as

$$\psi = \sqrt{b\nu}xF(\eta), \ \hat{p} = \hat{p}_w - \frac{1}{2}b\mu G(\eta), \ \eta = \sqrt{\frac{b}{\nu}}y \tag{3.8}$$

Stream function $\psi(x, y)$ in the form of velocity components is expressed as

$$\bar{u} = \frac{\partial\psi}{\partial y} = bxF'(\eta), \ \bar{v} = -\frac{\partial\psi}{\partial x} = -\sqrt{b\nu}F(\eta) \tag{3.9}$$

Here, $\nu = \frac{\mu}{\rho}$, the kinematic viscosity.

Utilizing Eq. (3.9) in the Continuity and momentum equations Eqs. (3.1) – (3.3), Continuity equation is valid and the dimensionless ordinary differential equation of momentum equations can be obtained as

$$F''' - F'^2 + FF'' - MF' = 0 \tag{3.10}$$

$$G' = 2(F'' + FF') \tag{3.11}$$

The transformed boundary conditions are

$$F(0) = 0,\ F'(0) = 1,\ G(0) = 0,\ F'(\eta \to \infty) \to 0 \tag{3.12}$$

Here, $M = \frac{\sigma B_o^2}{\rho b}$ is magnetic parameter. Let us assume the solution of Eq. (3.10) satisfying boundary conditions in Eq. (3.12).

$$F(\eta) = \frac{1}{\beta}(1 - e^{-\beta\eta}) \tag{3.13}$$

with $\beta = (1 + M)^{\frac{1}{2}}$. The non-dimensional pressure can directly be obtained by integrating Eq. (3.11)

$$G(\eta) = F^2 + 2F' - 2 \tag{3.14}$$

Now, utilizing Eqs. (3.13) and (3.14) in the velocity components and pressure, we obtain

$$\bar{u} = bxe^{-\beta\eta} \tag{3.15}$$

$$\bar{v} = -(b\nu)^{\frac{1}{2}}(1 - e^{-\beta\eta})/\beta \tag{3.16}$$

$$\hat{p} = \hat{p}_w - \frac{b\mu}{2\beta^2}(e^{-2\beta\eta} + 2(\beta^2 - 1)e^{-\beta\eta} + (1 - 2\beta^2)) \tag{3.17}$$

Here, Streamlines takes the form $\psi = \psi_o$ as

$$y = -(\frac{\nu}{b})^{1/2} ln[1 - \beta\psi_o(b\nu)^{-1/2}x^{-1}]/\beta \tag{3.18}$$

and the skin friction coefficient is obtained as

$$C_f = \frac{\mu\frac{d\bar{u}}{dy}|_{y=0}}{\rho\bar{u}_w^2/2} = \frac{2F''(0)}{Re_x^{1/2}} = -2\beta Re_x^{-1/2} \tag{3.19}$$

where, Reynolds number is $Re = \bar{u}_w x/\nu$. According to [4, 5] and [19], the velocity components remain identical throughout the boundary layer whereas the pressure distribution Eq. (3.17) could be vanished in this region. The boundary condition at $y \to \infty$ on the velocity components and pressure distribution will be

$$\bar{u} \to 0, \ \bar{v} \to -\frac{(b\nu)^{1/2}}{\beta}, \ p = p_w - c\mu\frac{(\frac{1}{2} - \beta^2)}{\beta^2} \tag{3.20}$$

The above conditions indicate that $\bar{u}$ - component of the velocity tends to zero and $\bar{v}$ - component of the velocity is treated as constant and the negative sign shows the downward direction. Nield [17] stated about the model of Brinkman equation in which he compared the saturated porous medium with uni-dimensional hydromagnetic fluid flow. Eqs. (3.15) and (3.16) are same to the problems of such medium in Liu [16] whereas the pressure between present literature and Liu [16] are not same due to which the validity of 2-D comparison fails.

3.2.2 Mass transfer problem

To obtain the non-dimensional concentration equation (3.4), we introduce a transformation as

$$\phi(\eta) = (\bar{C} - \bar{C}_\infty)/(\bar{C}_w - \bar{C}_\infty) \tag{3.21}$$

Applying the above transformations Eqs. (3.9), (3.13) and (3.21) in Eqs. (3.4), (3.6) and (3.7), we get

$$\frac{d^2\phi}{d\eta^2} + Sc\frac{(1 - e^{-\beta\eta})}{\beta}\frac{d\phi}{d\eta} + Sc(\alpha_1 - e^{-\beta\eta})\phi = 0 \tag{3.22}$$

with transformed B.Cs

$$\phi(\eta = 0) = 1, \;\; \phi \;\to\; 0 \quad \text{as} \quad \eta \to \infty \tag{3.23}$$

here, $Sc = \nu/D$ and $\alpha_1 = k_1/b$ are Schmidt number and reaction parameter, respectively. Assuming a new variable with $s = Sc/\beta^2$

$$\zeta = -se^{-\beta\eta}$$

Using the above assumption in Eqs. (3.22) – (3.23), we obtain the non-dimensional ODE

$$\zeta\frac{d^2\phi}{d\zeta^2} + (1 - s - \zeta)\frac{d\phi}{d\zeta} + (1 + s\alpha_1/\zeta)\phi = 0 \tag{3.24}$$

with

$$\phi(\zeta = -s) = 1, \ \ \phi \to 0 \quad \text{as} \quad \zeta \ \to \ 0^- \tag{3.25}$$

The solution of Eq. (3.24) is obtained in the standard Kummer's equation [2] as

$$\phi(\zeta) = (\frac{\zeta}{-s})^{a+b} M(a+b-1, 2b+1, \zeta)/M(a+b-1, 2b+1, -s) \tag{3.26}$$

where

$$a = s/2, \ \ b = (s^2 - 4\alpha_1 s)^{1/2}/2, \ \ M(p,q,t) = 1 + \sum_{m=1}^{\infty} \frac{(p)_m}{(q)_m} \frac{t^m}{m!}$$

is Kummer's function

$$(p)_m = p(p+1)(p+2)...(p+m-1), \ (q)_m = q(q+1)(q+2)...(q+m-1) \tag{3.27}$$

It is written in the form of η as

$$\phi(\eta) = e^{-\beta(a+b)\eta} M(a+b-1, 2b+1, -se^{-\beta\eta}/M(a+b-1, 2b+1, -s) \tag{3.28}$$

The local Nusselt number for concentration field at the sheet can be obtained as

$$J_w = -\rho D \frac{\partial \bar{C}}{\partial y}(0) = -\rho D B x (b/\nu)^{1/2} \phi'(0) \tag{3.29}$$

where

$$\phi'(\eta = 0) = -\beta(a+b) + \frac{a+b-1}{2b+1} s\beta \frac{M(a+b, 2b+2, -s)}{M(a+b-1, 2b+1, -s)} \tag{3.30}$$

3.3 Heat transfer problems

3.3.1 PST case

To get non-dimensional temperature equation can be obtained by using the transformation

$$\theta(\eta) = \frac{\bar{T} - \bar{T}_\infty}{\bar{T}_w - \bar{T}_\infty}$$

Utilizing above transformation, Eqs. (3.9) and (3.13) in Eqs. (3.5) – (3.7), we get

$$\frac{d^2\theta}{d\eta^2} + Pr \frac{(1 - e^{-\beta\eta})}{\beta} \frac{d\theta}{d\eta} + Pr(\alpha_2 - e^{-\beta\eta})\theta = 0 \tag{3.31}$$

and reduced boundary conditions

$$\theta(\eta = 0) = 1, \ \ \theta \ \rightarrow \ 0 \quad \text{as} \quad \eta \rightarrow \infty \tag{3.32}$$

where, $Pr = \mu c_p/k$ and $\alpha_2 = k_2/\rho c_p b$ are Prandtl number and internal heat parameter, respectively. It is seen that solution of Eq. (3.31) is similar to the solution of Eq. (3.22). Therefore its solution may directly be given as

$$\theta(\eta) = e^{-\beta(p+q)\eta} M(p+q-1, 2q+1, -re^{-\beta\eta})/M(p+q-1, 2q+1, -r) \tag{3.33}$$

for

$$r = Pr/\beta^2 \ , \ p = r/2 \ , \ q = (r^2 - 4\alpha_2 r)^{1/2}/2 \tag{3.34}$$

Nusselt number at given surface is developed from the rate of heat transfer, as

$$Nu_x = \frac{-k\bar{T}_y}{k(\bar{T}_w - \bar{T}_\infty)} x = -(\frac{bx^2}{\nu})^{1/2}\theta'(0) = Re_x^{1/2}\theta'(0) \tag{3.35}$$

and the non-dimensional temperature gradient is;

$$\theta'(\eta = 0) = -\beta(p+q) + \frac{p+q-1}{2q+1} r\beta \frac{M(p+q, 2q+2, -r)}{M(p+q-1, 2q+1, -r)} \tag{3.36}$$

3.3.2 PHF case

Here, non-dimensional temperature is considered to be

$$\bar{T} - \bar{T}_\infty = \frac{Ax}{k(b/\nu)^{1/2}}\Phi(\eta) \tag{3.37}$$

and temperature profile with B.Cs. are written as

$$\frac{d^2\Phi}{d\eta^2} + Pr\frac{(1-e^{-\beta\eta})}{\beta}\frac{d\Phi}{d\eta} + Pr(\alpha_2 - e^{-\beta\eta})\Phi = 0 \tag{3.38}$$

and

$$\Phi'(\eta = 0) = -1\ ,\ \Phi\ \rightarrow\ 0\ \ \text{at}\ \ \eta\ \rightarrow\ \infty \tag{3.39}$$

The above equation will yield a solution as

$$\Phi(\eta) = \frac{1}{\beta}\frac{e^{-\beta(p+q)\eta M(p+q-1,2q+1,-re^{-\beta\eta})}}{(p+q)M(p+q-1,2q+1,-r) - r\frac{p+q-1}{2q+1}M(p+q,2q+2,-r))} \tag{3.40}$$

here, surface temperature is

$$T = \bar{T}_\infty + \frac{Ax}{k}(\nu/b)^{1/2}\Phi(0) \tag{3.41}$$

where

$$\Phi(\eta = 0) = \frac{M(p+q-1,2q+1,-r)}{\beta(p+q)M(p+q-1,2q+1,-r) - r\beta\frac{p+q-1}{2q+1}M(p+q,2q+2,-r)} \tag{3.42}$$

3.4 Results and discussion

Velocity $[\bar{u}, \bar{v}, 0]$ and $\hat{p}$ are obtained using Anderson [13] for heat and mass transfer of hydromagnetic flow on stretching surface with uniform magnetic field to get the exact solution for heat transfer with its generation/absorption and for mass transfer using some chemical reaction of first order.

Due to application of the similarity variables in Eqs. (3.8) − (3.9), the momentum diffusion terms vanish and these terms are generally being neglected in boundary layer theory. A similarity variable for pressure in Eq. (3.8) is used to get the values of pressure gradient along y - axis (i.e.) $\frac{\partial p}{\partial y}$, the mass diffusion terms of the concentration and temperature profile are vanished by applying Eqs. (3.8) − (3.9). The validity of the concentration and temperature solutions lies with the following limitations.

$$\alpha_1 \leq \frac{Sc}{4(1+M)} \quad \text{and} \quad \alpha_2 \leq \frac{Pr}{4(1+M)}$$

If the values of M is increased, the range of above two parameters will be decreasing whereas the increase in the values of dimensionless numbers Pr and Sc will increase α_1 and α_2, respectively. One very interesting behavior is observed in these reactions that the validity of concentration solution is for the destructive reaction ($\alpha_1 < 0$) whereas the validity of temperature solutions is for internal heat absorption ($\alpha_2 < 0$) rather than generation.

To analyze the variation of velocity profile with magnetic parameter, Fig. (2) illustrates that increasing magnetic field decreases velocity in the field. In order to study the temperature at wall of the sheet or far from the sheet, it is observed the higher temperature at wall rather than far from it. Figs. (3) and (4) clearly depict non-dimensional temperature profiles $\theta(\eta)$ tend to zero for some of the physical parameters in both PST and PHF cases. Comparative study of curves with parameters assigned in both figures, curves increase with increasing values of M and α_2 and the internal heat parameter decreases with non-dimensional Prandtl number. However M has no direct interaction with temperature profile rather acts from the velocity profile due to which it increases the temperature profile indirectly. The two cases (PST and PHF) for the temperature field are being discussed to check the dependence of θ on α_2 because increasing value of α_2 results in implication of higher heat generation in the fluid so as the temperature on both cases may be increased.

In Figs. (5) and (6), temperature profile is variated for values of Prandtl number and is seen an increase in Prandtl number decreases temperature but one very interesting behavior is observed that the reduction of temperature in PST case is thickened on greater values of Pr and meets at unity whereas in PHF case it does not meet about unity rather disperses for greater values of Prandtl number.

Variations of concentration profile are also shown in the figures and change in profile is observed. Fig. (7) shows the increase in concentration by increasing magnetic parameter and Fig. (8) depicts the decrease in concentration profile by increasing values of Schmidt number. In Fig. (9), interesting behavior is observed that increasing reaction rate parameter α_1 increases concentration profile and huge difference is seen for $\alpha_1 > 0.9$.

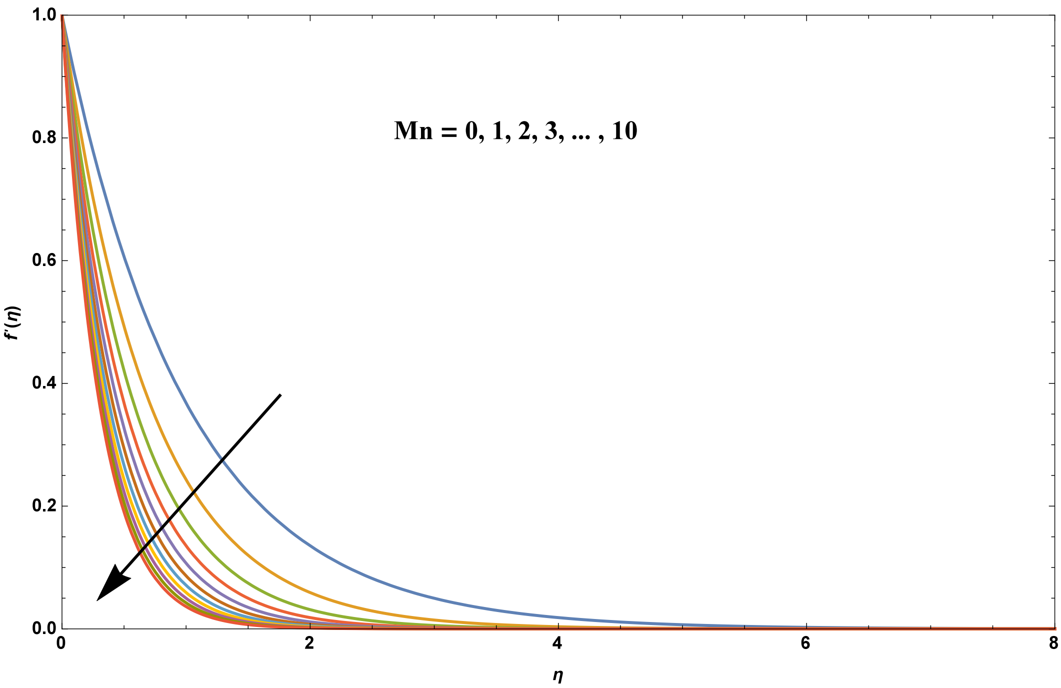


Fig. 2: Effects of magnetic paramter on velocity profile

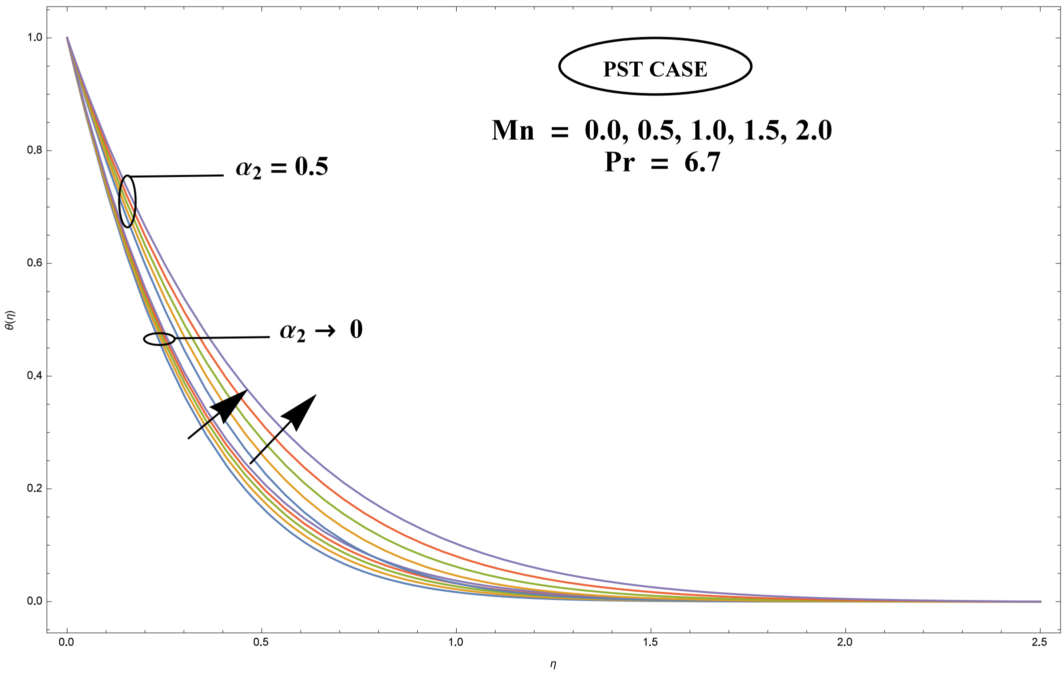


Fig. 3: Variation of temperature for selected parameters

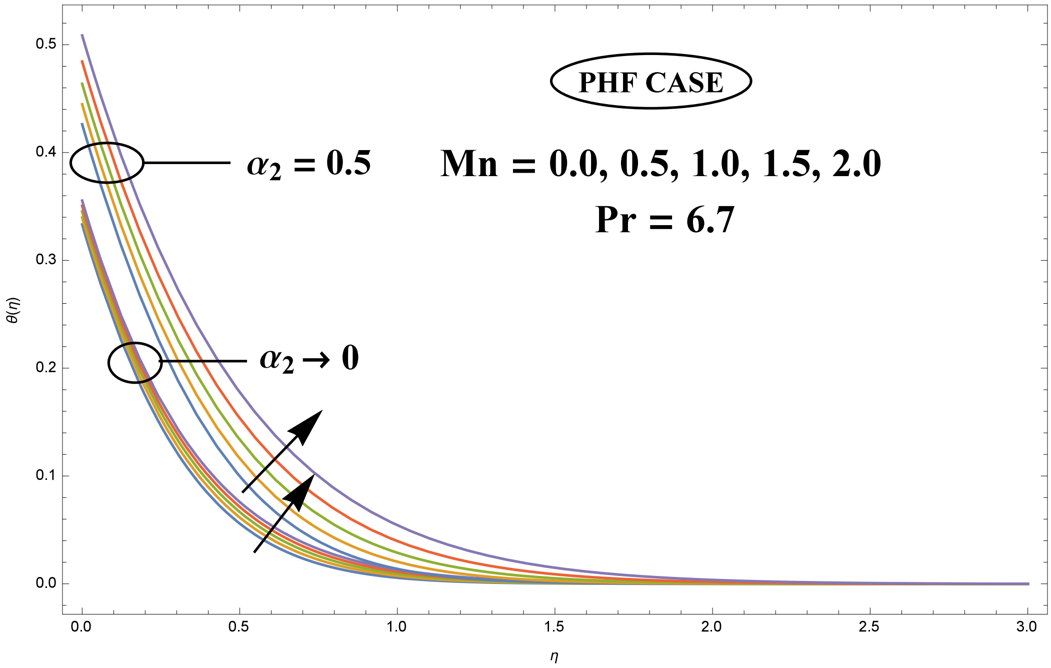


Fig. 4: Variation of temperature for selected parameters

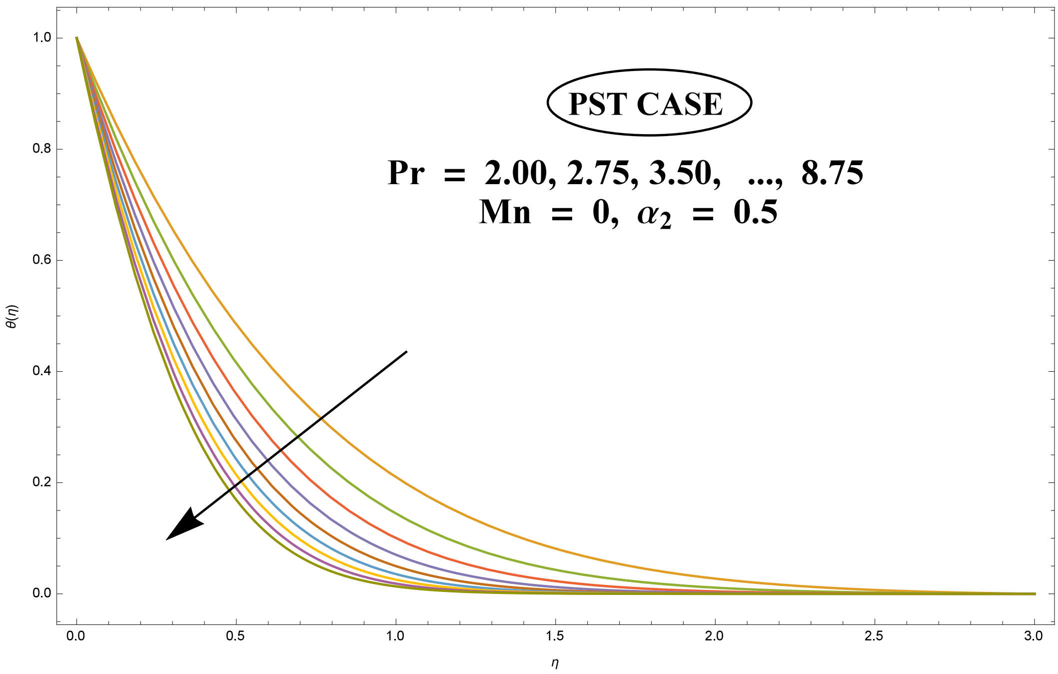


Fig. 5: Variation of Prandtl number on temperature profile

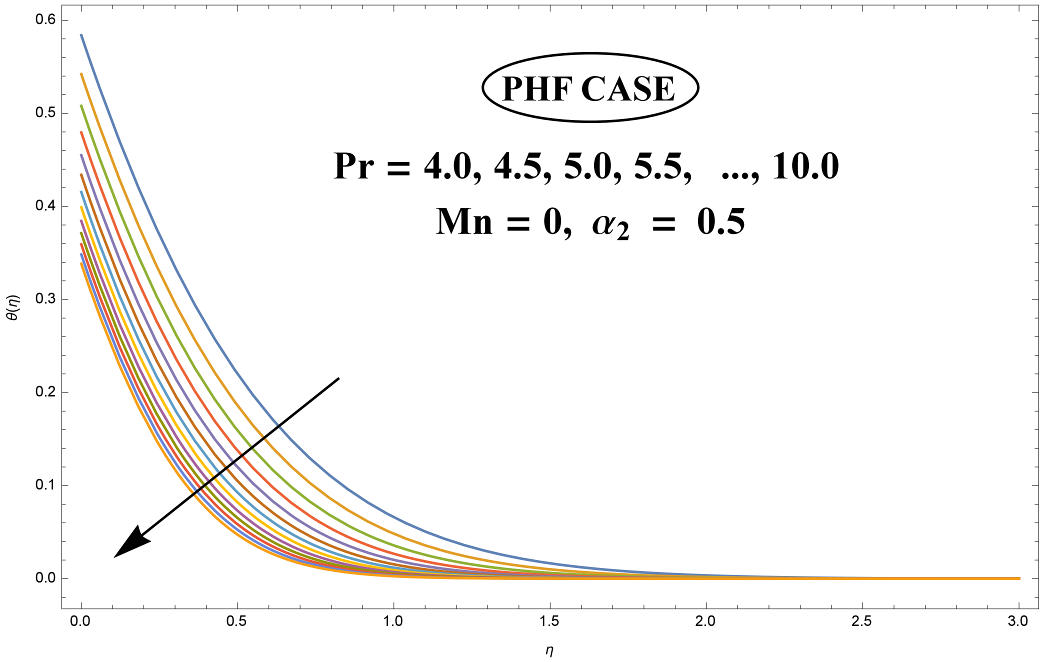


Fig. 6: Variation of Prandtl number on temperature profile

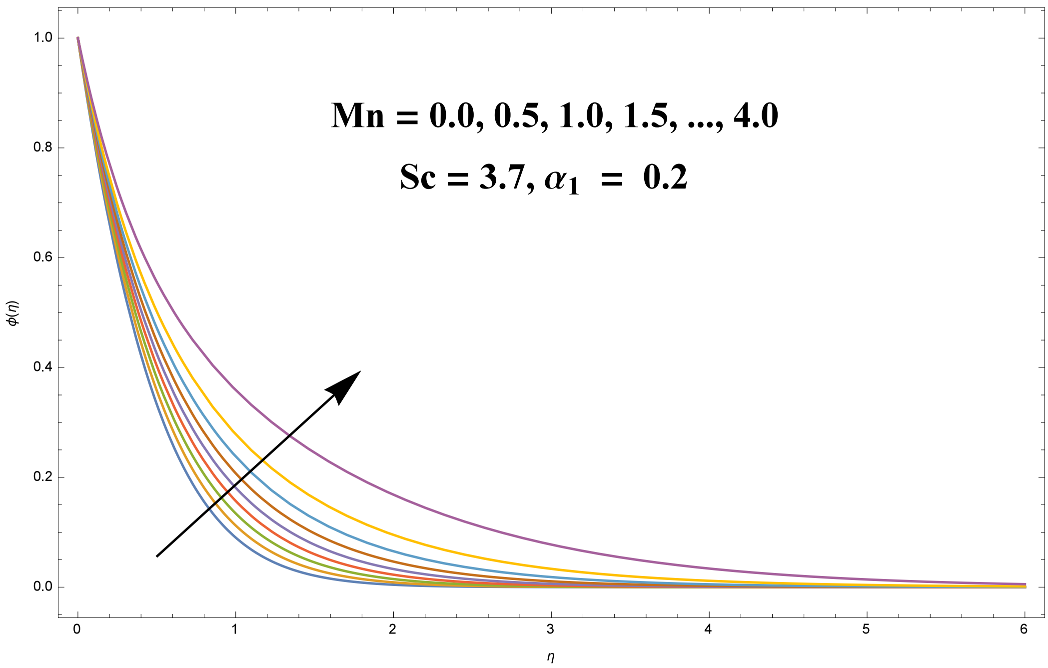


Fig. 7: Variation of magnetic parameter on concentration profile

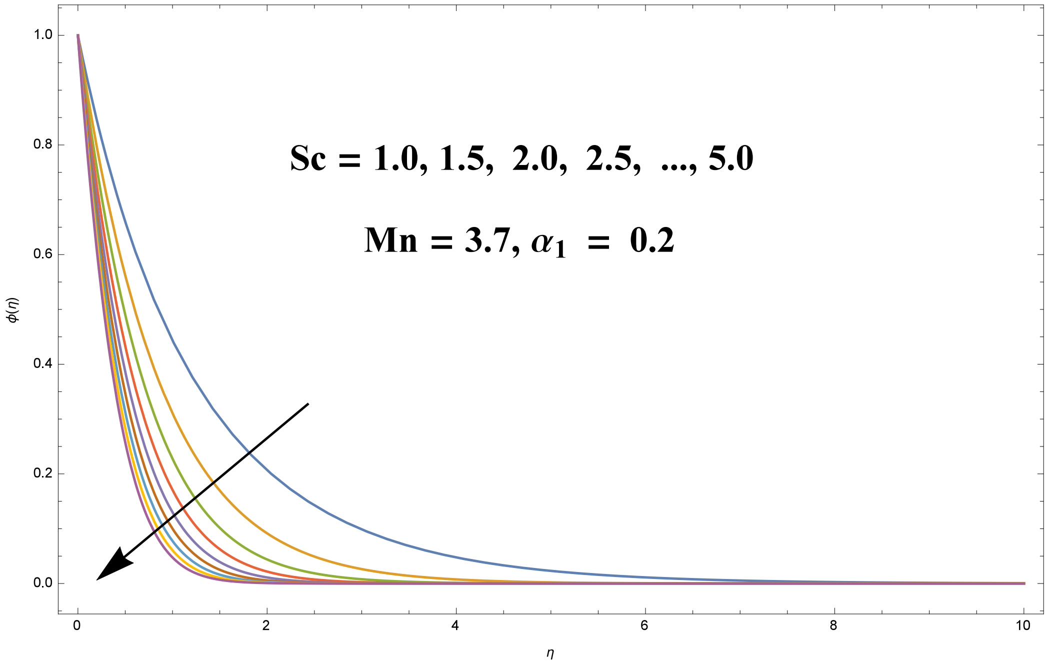


Fig. 8: Variation of Schmidt number on concentration profile

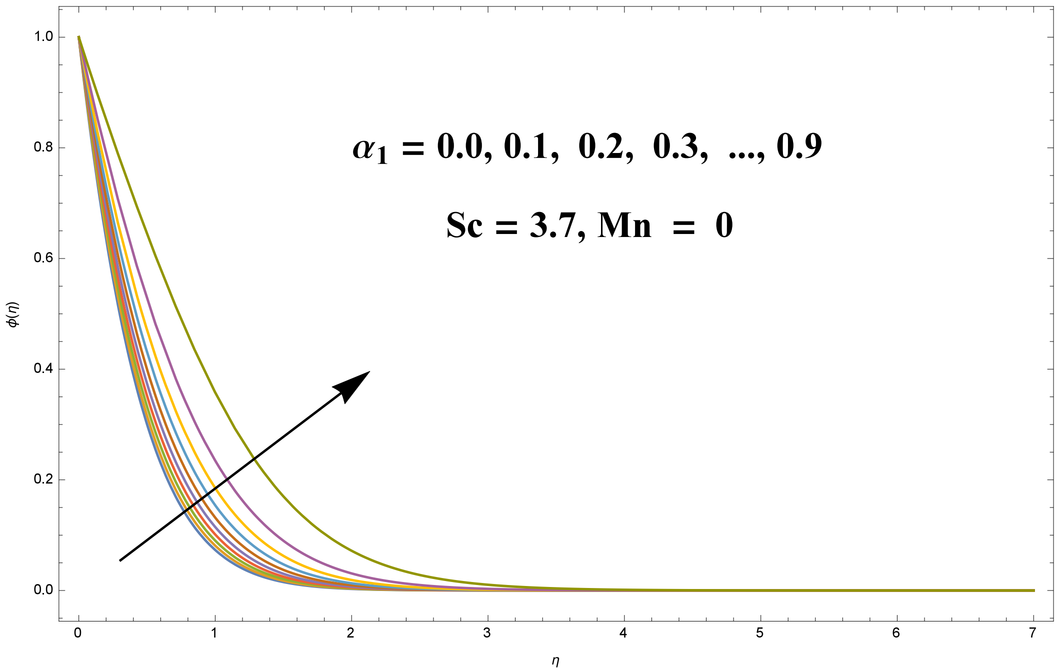


Fig. 9: Variation of reaction parameter on concentration profile

Chapter 4

Analysis of heat and mass transfer for Casson fluid along moving sheet in presence of thermal radiation and suction/injection effects

This chapter is based on description of the literature on heat and mass transfer. Mathematical modeling is developed that is based upon momentum, energy and concentration equation. Constraints are defined for boundary layer flow. Finally, a numerical method is applied to attain the stable solution of entire mathematical model and hence, determined the velocity, temperature and concentration profile.

4.1 Mathematical formulation

Considering an electrically conducting incompressible fluid passing through a non-conducting and flat impermeable stretched sheet $y = 0$ having a velocity bx for positive b and x are set horizontal in coordinate plane. The fluid flows on $y > 0$ under the consideration of uniform B_o. On the fixed origin, the coupled forces are applied to stretch the sheet horizontally elsewhere the quiescent fluid's motion is just driven by stretching surface. The rheological equation for an isotropic flow [6] is defined as

$$\begin{aligned} \bar{\tau}_{ij} &= 2(\mu_B + \bar{p}_y/\sqrt{2\bar{\pi}})\bar{e}_{ij} \quad , \quad \bar{\pi} > \bar{\pi}_c \\ &= 2(\mu_B + \bar{p}_y/\sqrt{2\bar{\pi}_c})\bar{e}_{ij} \ , \quad \bar{\pi} < \bar{\pi}_c \end{aligned}$$

where $\bar{\pi} = \bar{e}_{ij}\bar{e}_{ij}$ is the (i, j)th component of the deformation rate, $\bar{\pi}$ is its product with itself, $\bar{\pi}_c$ is the critical value of this product based on the non-

Newtonian model, μ_B is plastic dynamic viscosity, and $\bar{p}_y$ is yielded stress of the fluid.

The governing equations of 2-D Casson fluid are expressed as

$$\bar{u}_x + \bar{v}_y = 0, \tag{4.1}$$

$$\rho(\bar{u}\bar{u}_x + \bar{v}\bar{u}_y) = -\frac{\partial \hat{p}}{\partial x} + \mu(1+\frac{1}{\gamma})(\bar{u}_{xx} + \bar{u}_{yy}) + \bar{u}_e\frac{d\bar{u}_e}{dx} - \sigma B_o^2(\bar{u} - \bar{u}_e), \tag{4.2}$$

$$\rho(\bar{u}\bar{v}_x + \bar{v}\bar{v}_y) = -\frac{\partial \hat{p}}{\partial y} + \mu(1+\frac{1}{\gamma})(\bar{v}_{xx} + \bar{v}_{yy}), \tag{4.3}$$

$$\bar{u}\bar{C}_x + \bar{v}\bar{C}_y = D(\bar{C}_{xx} + \bar{C}_{yy}) + k_1(\bar{C} - \bar{C}_\infty), \tag{4.4}$$

$$\rho c_p(\bar{u}\bar{T}_x + \bar{v}\bar{T}_y) = k(\bar{T}_{xx} + \bar{T}_{yy}) + k_2(\bar{T} - \bar{T}_\infty) - \frac{\partial q_r}{\partial y}. \tag{4.5}$$

where, $\gamma = \mu_B\sqrt{2\bar{\pi}_c}/\bar{p}_y$ is Casson parameter of non-Newtonian fluid. Smaller value of induced magnetic field is obtain on neglecting electric field as compare to applied magnetic field which results $-\sigma B_o^2 u$ from the assumption in the case of magnetic field normal to velocity field. In Eqs. (4.4) and (4.5), the positive values of k_1 and k_2 represent constructive chemical reaction of species and internal heat generation, respectively whereas the negative values of k_1 and k_2 are the destructive chemical reaction and internal heat absorption, respectively. It is observed above that temperature depends upon k_1 while k_1 is assumed to be constant by [16].

q_r in Eq. (4.5) can be written using Rosseland approximation [3] as

$$q_r = -\frac{4\sigma^*}{3k^*}\frac{\partial \bar{T}^4}{\partial y},$$

Here $\bar{T}^4$ can be expanded through Taylor's series by neglecting its higher orders, we can get as $\bar{T}^4 \equiv 4\bar{T}_\infty^3\bar{T} - 3\bar{T}_\infty^4$. Therefore, Eq. (4.5) can be written as

$$\rho c_p(\bar{u}\bar{T}_x+\bar{v}\bar{T}_y) = k(\bar{T}_{xx}+\bar{T}_{yy})+k_2(\bar{T}-\bar{T}_\infty)+\frac{16\sigma^*\bar{T}_\infty^3}{3k^*}\bar{T}_{yy}. \tag{4.6}$$

The boundary conditions are defined as

$$\bar{u} = \bar{u}_w = bx, \;\; \bar{v} = \bar{v}_w = cx, \;\; p = p_w, \;\; \bar{C} = \bar{C}_w = \bar{C}_\infty+Bx,$$

$$\text{PST:} \quad \bar{T} = \bar{T}_w = \bar{T}_\infty + Ex,$$
$$\text{PHF:} \;\; q_w = -k\frac{\partial \bar{T}}{\partial y} = Ax \quad \text{at} \quad y \to 0. \tag{4.7}$$
$$\bar{u} \to \bar{u}_e = ax, \; \bar{C} \to \bar{C}_\infty, \; \bar{T} \to \bar{T}_\infty \;\; \text{at} \;\; y \to \infty. \tag{4.8}$$

Transformation to calculate numerical solution of given boundary problem of hydromagnetic fluid can be introduced as

$$\psi = \sqrt{b\nu}xF(\eta), \; p = p_w-\frac{1}{2}b\mu G(\eta), \; \eta = \sqrt{\frac{b}{\nu}}y. \tag{4.9}$$

Stream function in the form of velocity components are expressed as

$$\bar{u} = \frac{\partial\psi}{\partial y} = bxF'(\eta), \; \bar{v} = -\frac{\partial\psi}{\partial x} = -\sqrt{b\nu}F(\eta), \tag{4.10}$$

$$\phi(\eta) = (\bar{C}-\bar{C}_\infty)/(\bar{C}_w-\bar{C}_\infty), \tag{4.11}$$

PST case:

$$\theta(\eta) = (\bar{T} - \bar{T}_\infty)/(\bar{T}_w - \bar{T}_\infty). \tag{4.12}$$

PHF case:

$$\bar{T} - \bar{T}_\infty = \frac{Ax}{k(b/\nu)^{1/2}}\Phi(\eta). \tag{4.13}$$

Here, $\nu = \frac{\mu}{\rho}$, the kinematic viscosity.

Utilizing Eq. $(4.10) - (4.12)$ in Continuity, momentum, concentration and energy equations $(4.1) - (4.4)$ and (4.6), Continuity equation is satisfied and the dimensionless ordinary differential equation of momentum, concentration and energy equations for F, G, ϕ and θ can be obtained as;

$$(1 + \frac{1}{\gamma})F''' - F'^2 + FF'' + M(r - F') + r^2 = 0, \tag{4.14}$$

$$G' = 2(1 + \frac{1}{\gamma})F'' + 2FF', \tag{4.15}$$

$$\phi'' + Sc\ F\ \phi' + Sc(\alpha_1 - F')\phi = 0, \tag{4.16}$$

$$(1 + \frac{4R}{3})\theta'' + Pr\ F\ \theta' + Pr(\alpha_2 - F')\theta = 0. \tag{4.17}$$

The transformed B.Cs are

$$F(0) = -S,\ F'(0) = 1,\ G(0) = 0,$$

$$F' \to r \quad \text{as} \quad \eta \to \infty, \tag{4.18}$$

$$\phi(\eta = 0) = 1,\ \phi \to 0 \ \text{ as } \ \eta \to \infty, \tag{4.19}$$

$$\text{PST case: } \theta(\eta = 0) = 1,\ \theta \to 0 \quad \text{at} \quad \eta \to \infty, \tag{4.20}$$

PHF case:

$$\Phi'(\eta = 0) = -1\ ,\ \Phi\ \rightarrow\ 0\ \text{ at } \eta\ \rightarrow\ \infty. \tag{4.21}$$

For $S > 0$ is suction and $S < 0$ is an injection parameter. $M = \frac{\sigma B_o^2}{\rho b}$, magnetic parameter with $\beta = (1 + M)^{\frac{1}{2}}$. $\alpha_1 = k_1/b$, reaction parameter, $Pr = \mu c_p/k$, Prandtl number, $\alpha_2 = k_2/\rho c_p b$, internal heat parameter and $r = \frac{a}{b}$ is the stagnation point.

4.2 Numerical solution

To solve the nonlinear ODEs Eqs. (4.14) – (4.17) numerically by shooting method. We assume the functions as;

$$F = y_1, \quad F' = y_2, \quad F'' = y_3,$$

$$F''' = y_3' = (y_2^2 - y_1 y_3 - M(r - y_2) - r^2)/(1 + \frac{1}{\gamma}), \tag{4.22}$$

$$G = y_4, \quad G' = y_4' = 2(y_1 y_2 + y_3), \tag{4.23}$$

$$\theta = y_5, \quad \theta' = y_6,$$

$$\theta'' = y_6' = (-Pr\frac{(1 - e^{-\beta\eta})}{\beta} y_8 - Pr(\alpha_2 - e^{-\beta\eta}) y_7)/(1 + \frac{4}{3}R), \tag{4.24}$$

$$\phi = y_7, \quad \phi' = y_8,$$

$$\phi'' = y_8' = -Sc\frac{(1 - e^{-\beta\eta})}{\beta} y_{11} - Sc(\alpha_1 - e^{-\beta\eta}) y_{10}. \tag{4.25}$$

The transformed boundary conditions in Eqs. (4.18) – (4.21) become

$$y_1(0) = -S,\ y_2(0) = 1,\ y_4(0) = 0,\ y_2(\eta \to \infty) \to r, \tag{4.26}$$

PST Case:

$$y_5(0) = 1, \quad y_5(\eta \to \infty) \to 0, \tag{4.27}$$

PHF Case:

$$y_5(0) = -1, \quad y_5(\eta \to \infty) \to 0, \tag{4.28}$$

$$y_7(0) = 1, \quad y_7(\eta \to \infty) \to 0 \tag{4.29}$$

The above nonlinear ODEs are solved by using the bvp4c of MATLAB to get the solutions graphically.

4.3 Results and discussion

In order to obtain required results of above boundary value problems. we first transformed all the PDEs of momentum, energy and concentration into nonlinear ODEs by similarity transformation with their corresponding boundary conditions. One very interesting behavior is observed that the energy equation is being transformed into two case: The PST case and The PHF case. The results for both of these cases are shown independently through graphs. The nonlinear ODEs are illustrated for the velocity, temperature and concentration profiles and the effects are seen with all physical parameters involved.

The effects of some of physical parameters are seen in velocity profile where the magnetic parameter, stagnation point and Casson fluid are depicted and interesting behavior is seen. Fig. (10) illustrates effect of stagnation point on velocity distribution in which velocity increases with increasing stagnation point.

In Figs. (11) and (12), it is observed that velocity decreases with an increase in magnetic parameter and Casson fluid. One more thing is observed while applying Casson fluid, γ on velocity that for very large values of γ, the velocity distribution gives a straight line, consequently.

Figs. (13a) and (13b) illustrate the effects of temperature distribution for PST and PHF cases. It is seen that increasing values of magnetic parameter, M for $\mathrm{Pr} = 6.7$, thermal radiation and internal heat parameter results in increasing of temperature distribution in both PST and PHF cases. Interestingly, temperature distribution for PST case meets at unity whereas it does not meet at unity for PHF case rather disperses on greater values. This behavior is observed for all temperature profiles on different physical parameters.

In Figs. (14a) and (14b), temperature decreases with increase Prandtl number for PST and PHF cases with fixed internal heat parameter, $\alpha_2 = 0.1$. Same behavior is observed as previous Figs. (14a) and (14b) that temperature converges at unity in PST case but not in PHF. This is because of adiabatic temperature in the boundary condition. This result is seen at a particular point of the flow.

Figs. (15a) and (15b) depict the variation of temperature for the values thermal radiation. Increase in thermal radiation results an increase in temperature profile. The adiabatic temperature in boundary condition has an effect on temperature distribution for the thermal radiation and convergence at unity is again seen in PST case except on the PHF case.

In Figs. (16a) and (16b), the variation of temperature profile is analyzed for magnetic parameter at internal heat parameter tends to zero i.e. $\alpha_2 \to 0$. Increasing magnetic parameter increases temperature distribution. Interestingly, the effect of all parameters on temperature depict convergence at unity in PST case but not in PHF case and one more thing is observed here that temperature distribution is qualitatively same in PST and PHF cases whereas quantitatively different.

Figs. (17) and (18) depict the variations of M and Sc on concentration profile. Concentration increases with increase in magnetic parameter whereas decreases with the increase in the values of Schmidt number. Fig. (19) illustrates the effects of suction parameter on velocity profile where increasing suction parameter increases velocity distribution.

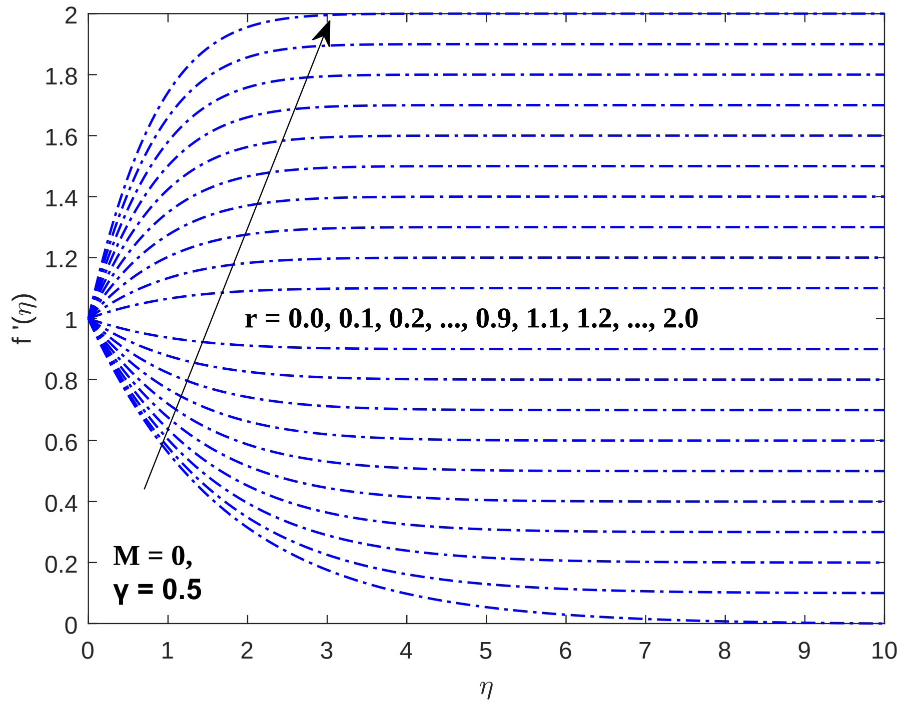


Fig. 10: Variations at stagnation point, r

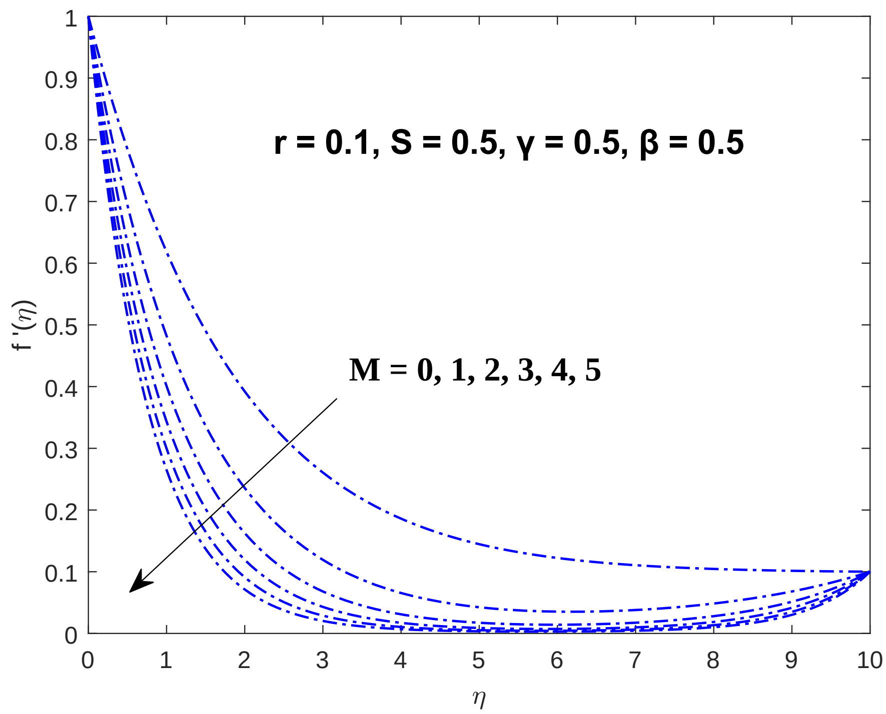


Fig. 11: Effects of magnetic parameter on velocity profile

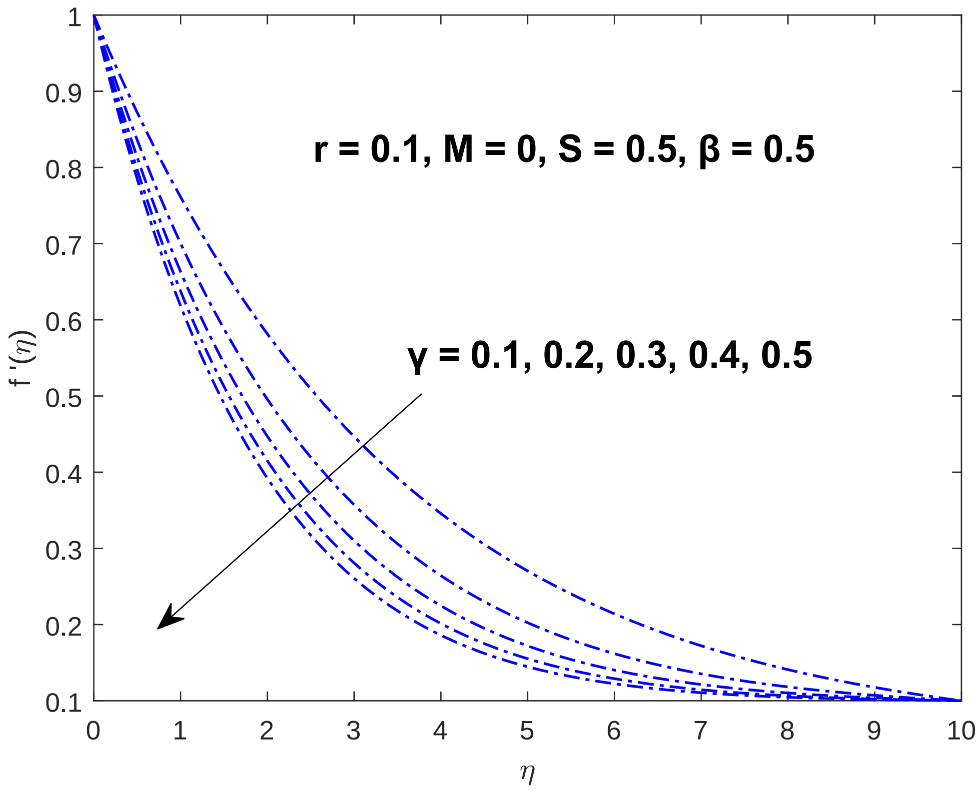
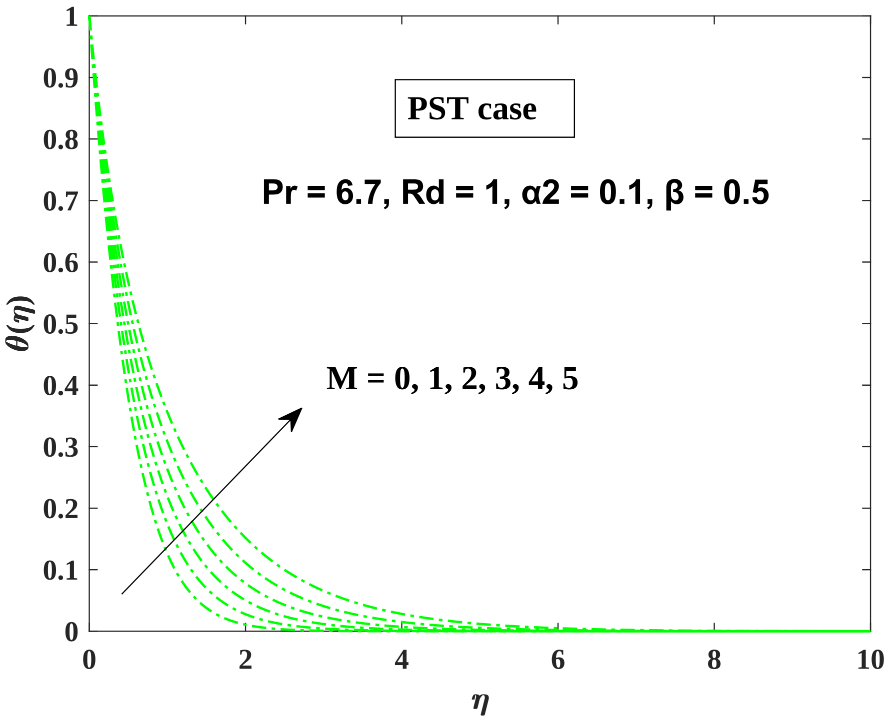
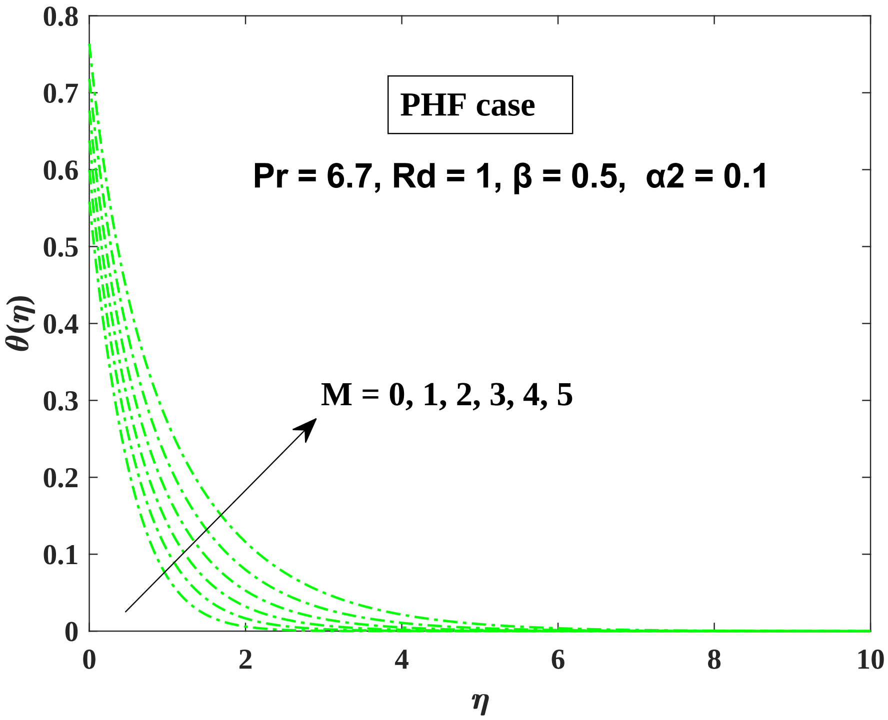


Fig. 12: Effects of Casson fluid parameter on velocity profile

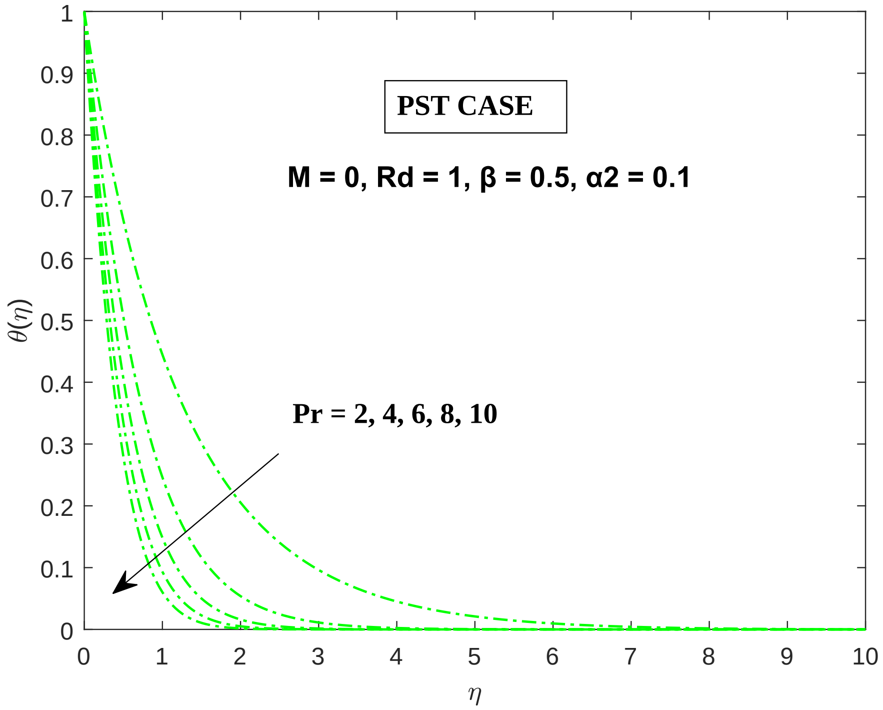


(a)

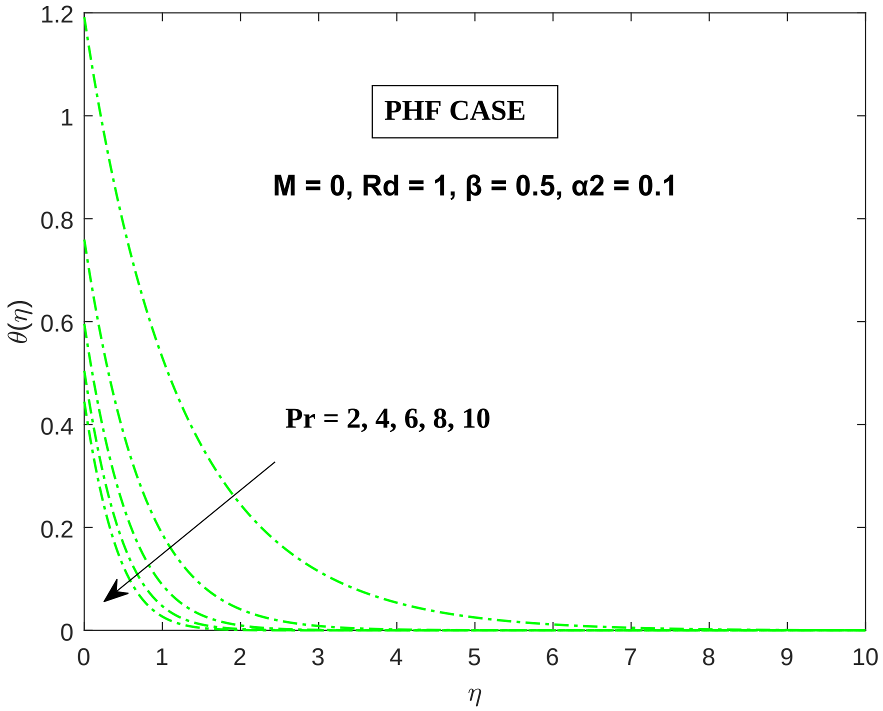


(b)

Fig. 13: Effects of magnetic paramter, M on temperature profile

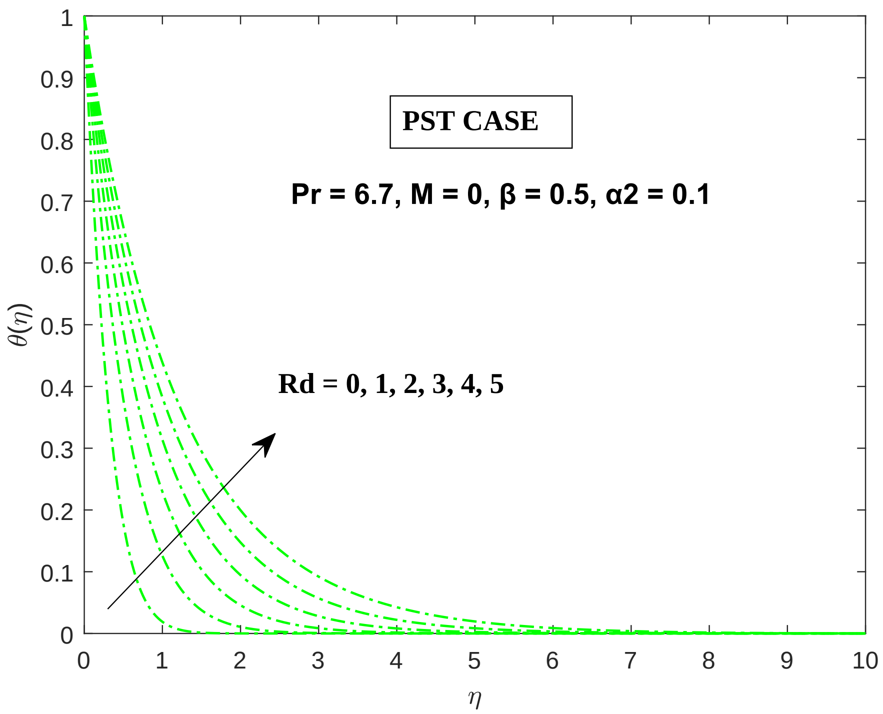


(a)

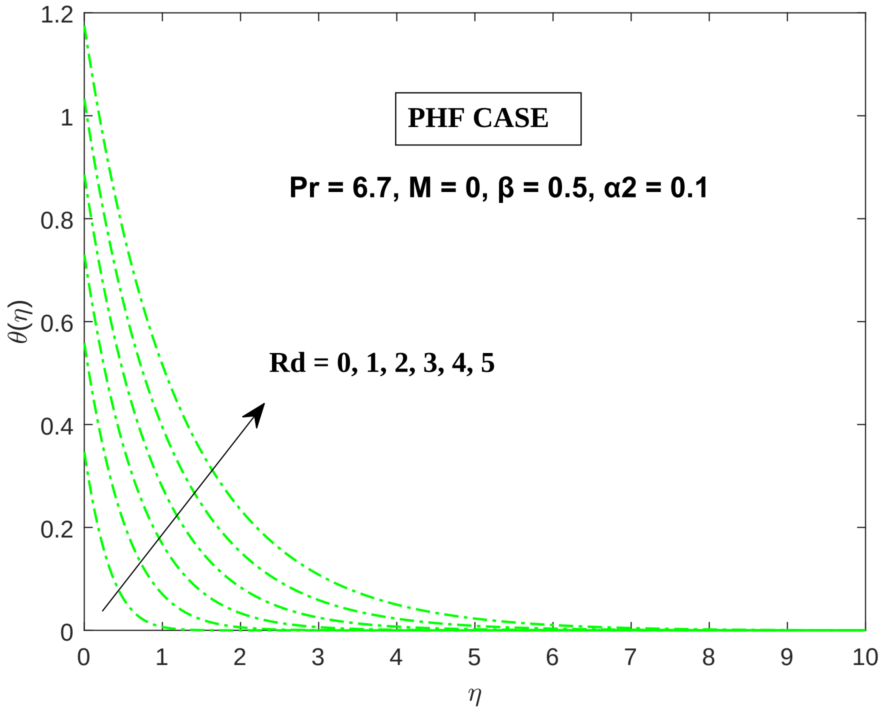


(b)

Fig. 14: Effects of Prandtl number, Pr on temperature profile

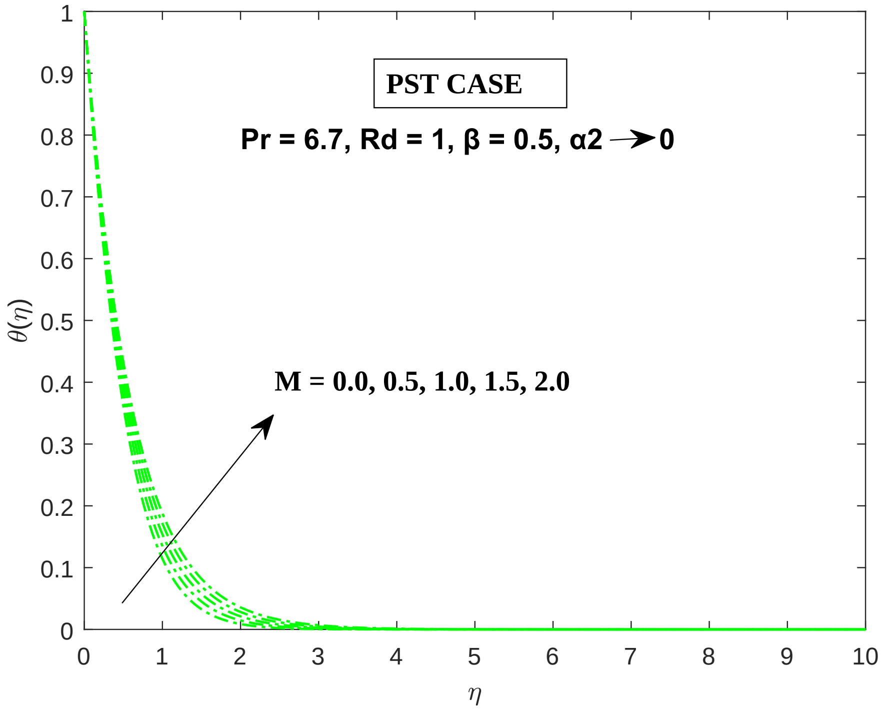


(a)

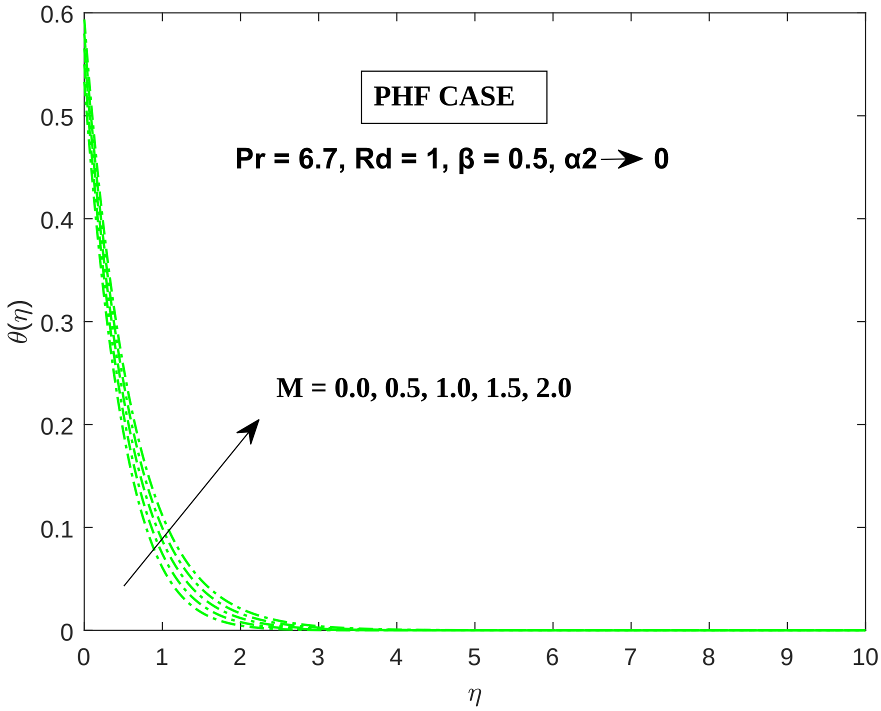


(b)

Fig. 15: Effects of Rd on temperature profile



(a)



(b)

Fig. 16: Variation of M for α_2 on temperature profile

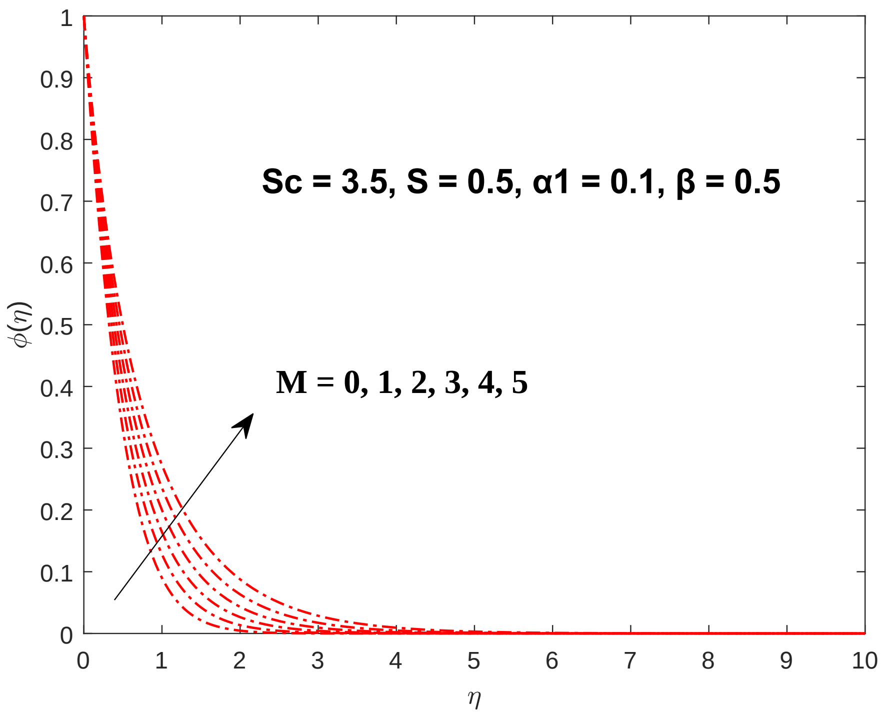


Fig. 17: Variation of M on concentration profile

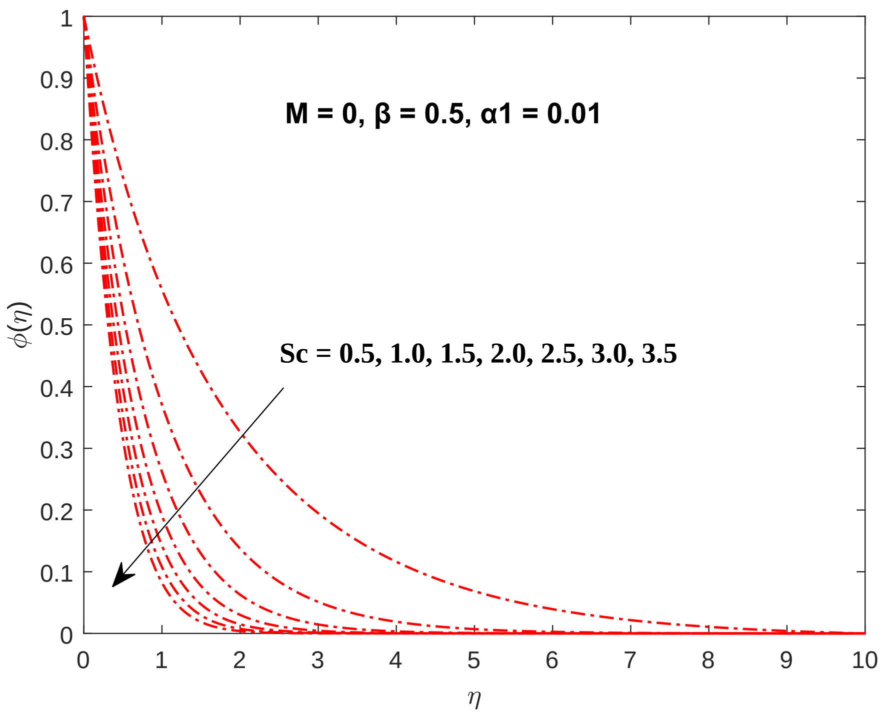


Fig. 18: Variation of Sc on concentration profile

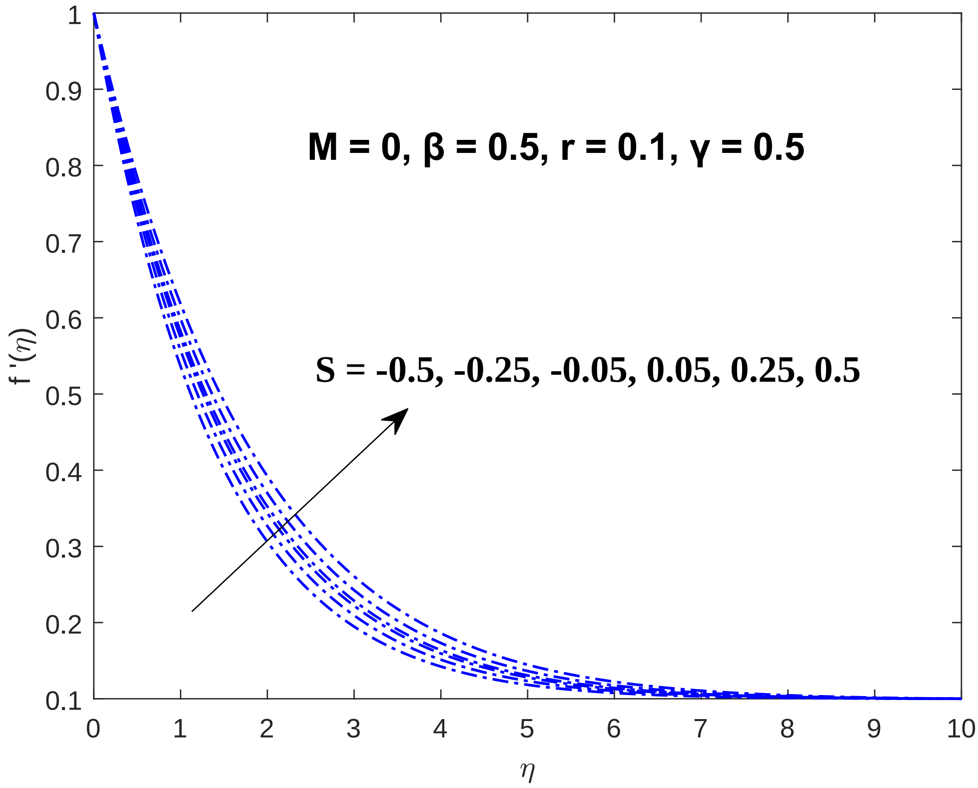


Fig. 19: Effects of suction parameter on velocity profile

Chapter 5

Conclusion

Initially, an article was reviewed and formulated its mathematical equations. Later, equations are solved analytically to get its closed form solution. The parameters obtained from these equations were shown graphically using Mathematica. At the end, a thorough discussion on results and graphs are upheld.

Addition of some new parameters in review article took place in an extension work where mathematical equations have been passed through numerical solution of nonlinear ODEs and are analyzed for heat and mass transfer of Casson fluid with thermal radiation and magnetic field. The following points are noted from the above results.

- Increasing magnetic parameter decreases velocity distribution.
- Two cases (PST and PHF) are analyzed for variable values of α_2 in which increasing magnetic parameter causes an increase in temperature distribution.
- For both PST and PHF case, increase in Pr decreases temperature distribution.
- Different behavior is observed in variation of concentration profile for M, Sc and α_1.
- Velocity profile increases on the stagnation point and magnetic parameter whereas decreases for the Casson fluid.
- Interesting behavior is seen in the temperature profile with its both cases (PST and PHF) on magnetic parameter, Prandtl number, internal heat parameter and thermal radiation.

- Magnetic parameter and thermal radiation increase the temperature distribution for $Pr = 6.7$ and $\alpha_2 \to 0, \alpha_2 = 0$ whereas Prandtl number decrease temperature profile.
- Effects of the magnetic parameter and Schmidt number are also seen in the concentration profile.
- Magnetic parameter increases concentration whereas Schmidt number decreases.

References

[1] T. Sarpkaya, Flow of non-Newtonian fluids in a magnetic field, AlChE J., 7 (1961), 324.

[2] M. Abramowitz, I.A. Stegun, Handbook of Mathematical Functions, Dover, New York, 1965.

[3] M.Q. Brewster, Thermal radiative transfer properties, John Wiley and Sons (1972).

[4] K. B. Pavlov, Magnetohydrodynamic flow of an incompressible viscous fluid caused by deformation of a plane surface, 10 (1974), 146-148.

[5] A. Chakrabarti, A.S. Gupta, Q. Appl. Math., 37 (1979), 73.

[6] M. Nakamura, T. Sawada, ASME J. Biomech. Eng., 110 (1988), 137.

[7] C.Y. Wang, Exact solutions of the steady-state Navier-Stokes equations, Annu. Rev. Fluid Mech., 23 (1991), 159 – 177.

[8] H.I. Andersson, Acta Mech., 95 (1992), 227.

[9] D. Rollins, K. Vajravelu, Int. J. Nonlinear Mech., 27 (1992), 265.

[10] M.I. Char, J. Math. Anal. Appl., 186 (1994), 674.

[11] R.F. Probstein, Physicochemical Hydrodynamics, John Wiley and sons, New York, 1994.

[12] H.I. Andersson, O.R. Hansen, B. Holmedal, Int. J. Heat Mass transfer, 37 (1994), 659.

[13] H.I. Andersson, an exact solution of Navier Stokes equations for MHD flow, 113 (1995), 241 – 244.

[14] E.M.A. Elbashbeshy, Radiation effect on heat transfer over a stretching surface, Can. J. Phys., 78 (2000), 1107 - 1112.

[15] S. Abel, P.H. Veena, K. Rajgopal, V.K. Pravin, Int. J. Nonlinear Mech., 39 (2004), 1067.

[16] I.C. Liu, Int. J. Heat Mass Transfer, 47 (2004), 4427.

[17] D.A. Nield, A. Bejan, Convection in Porous Media, 3rd ed. Springer, New York, 2006.

[18] Chhabra RP, Richardson JF, Non-Newtonian flow and applied rheology. 2nd edn. Butterworth-Heinemann, Oxford (2008).

[19] E. Abu-Nada, Application of nanofluids for heat transfer enhancement of separated flows encountered in a backward facing step, Int. J. Heat Fluid Flow, 29 (2008), 242-249.

[20] S.M.M. El-Kabeir, A. Chamkha, A.M. Rashad, Heat and mass transfer by MHD stagnation point flow of a power law fluid towards a stretching surface with radiation, chemical reaction and Soret and Dufour effects, Int. J. Chem. React. Eng., 8 (2010), 132.

[21] J.A. Khan, M. Mustafa, A numerical analysis for non-linear radiation in MHD flow around a cylindrical surface with chemically reactive species, Results in Physics, 8 (2018), 963 - 970.

[22] Khan, I., Ullah, S., Malik, M.Y., Hussain, A., Numerical analysis of MHD Carreau fluid flow over a stretching cylinder with Homogenous-heterogeneous reactions, Results in Physics (2018).

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