Dual nature study of convective heat transfers of nanofluid along a shrinking surface in a porous medium with thermal radiation

Thesis Submitted By

ALI RAZA

01-248172-001

Supervised By

Dr. Rizwan ul Haq

A dissertation submitted to the Department of Computer Science, Bahria University, Islamabad as a partial fulfillment of the requirements for the award of the degree of MS

Session (2017 - 2019)

Dedicated to

My worthy parents and respected teachers

whose prays and support have always been a source of inspiration

and encouragement for me

My caring and supporting wife and lovely daughters

have always given me care and love.

Acknowledgments

I am thankful to Almighty **ALLAH** Who has enabled me to learn and to achieve milestones towards my destination and His beloved Prophet Hazrat **Muhammad** (ﷺ) Who is forever a constant source of guidance, a source of knowledge and blessing for entire creation. His teachings show us a way to live with dignity, stand with honor and learn to be humble.

My acknowledgment is to my kind, diligent and highly zealous supervisor, Dr. Rizwan ul Haq, who supported me with his cherished opinions and inspirational discussions. His valuable expertise, comments, suggestions and instructions are most welcome that greatly improved the clarity of this document. I am placing my earnest thanks to Dr. Rizwan ul Haq. I am so grateful to work under the supervision of such a great person.

My gratitude is to my honorable professors who took me to the apex of my academia with their guidance. In particular, Prof. Dr. Muhammad Ramzan and Dr. Jafar Hasnain who have always been supportive in all of my course work and kept encouraging me throughout the session in Bahria University, Islamabad Campus. They are the true teachers who have made Mathematics Department of BUIC, a real place of learning.

My intense recognition is to my mother, brothers, sisters and wife (for every thing) who are always real pillars for my encouragement and showered their everlasting love, care and support throughout my life. Humble prayers, continuing support and encouragement of my family are as always highly appreciated.

As usual, so many friends and my class-mates have helped me throughout my MS that I cannot list them all. In particular, Dr. Abid Majeed, Hamid Shehzad, Shahid Iqbal, Muhammad Zubair, Umar Ali Shah, Syed Saqib Shah, Syed Hammad Shah, Adeela Mubeen and Salma Batool were

specially remained enormously helpful throughout the period of my MS studies.

Consequently, My all plea is to **Allah**, the Almighty, the beneficent Whose blessings are always showered upon me via strengthening my wisdom and bestowed me with the knowledge of what he wants.

Ali Raza

Bahria University Islamabad, Pakistan

June, 2019

Contents

List of Figures

List of Tables

Nomenclature

V	velocity vector
ρ	fluid's density
T	fluid's temperature
τ	Cauchy stress tensor
μ_{eff}	effective viscosity
μ	dynamic viscosity
ρC_p	heat capacity
k_{nf}	effective thermal conductivity of nanofluid
$\bar{v}_w$	surface mass transfer
$\bar{h}_f$	heat transfer coefficient
$\bar{T}_f$	convective fluid temperature
$\bar{T}_\infty$	temperature at infinity
a, b	positive constant numbers
K	permeability of porous medium
C_f	skin friction coefficient
Nu	local Nusselt number
$\bar{\tau}_w$	skin friction at wall
$\bar{q}_w$	heat transfer
α	stretching parameter
Ξ	viscosity ratio parameter
Pr	Prandtl number
Φ	porous medium parameter
Bi	Biot number
S	mass suction parameter
Re	Reynolds number
ν	kinematic viscosity
ϕ	nanofluid volume fraction

Abstract

The aim of this thesis is to express whole concept of analytic dual nature exact solutions. The thorough study of the heat transfers of nanoparticles of fluid along a shrinking surface in the presence of thermal radiation is analyzed. Initially, mathematical formulation is extracted that is consisting of momentum and energy equations. Later on, these PDEs of momentum and energy are converted in the dimensionless non-linear ODEs by applying some suitable similarity variables and the boundary conditions are defined with the thermal radiation term. Then the acquired solution of the dimensionless nonlinear ODEs will be evaluated by incomplete gamma solution, analytically. The results obtained through solutions for different physical parameters, namely, mass suction, stretching, porous medium, viscosity ratio, Prandtl number, Biot number, nanoparticle volume friction and thermal conductivity on velocity and temperature profiles are analyzed and elaborated graphically. Graphical presentation of the skin friction and Nusselt number also depicted.

Chapter 1

Introduction and literature review

Medium containing a minute opening in the surface that allows passage of fluids through these holes is called porous medium. The objects consisting of the porous surfaces have void empty spaces or pores through which fluid particles penetrate the object. It exists in a nature as sands, limestone, wooden materials, tissue papers, human lungs and biological tissues. There are many man-made materials containing pores like cements, foams, sponges and many more. Electric conduction is one of the property of media whose deduction tends to complexity. This media has widely been discussed in applied sciences (such as geosciences, biology and biophysics and material sciences) and engineering: filtration, mechanics. The flow under consideration is being studied by *Nield and Bejan* [7] and some of the literature is also being found in *Pop and Ingham* [6] comprehensively.

Flow around the surface is an essential concept in Fluid mechanics that raises in the domain of the such surfaces in which the viscous result is important. Governing equation of the Navier Stokes has greatly been made easier to understand with assumptions of the layer in the immediate vicinity of the fluid flow. In recent times, the flow on the boundary of fluid with the transfer of heat is enormously engaged researchers due to its several consequential utilizations of expulsion of liquefied polymer via slit die, glass wares, paper making, metal spinning and artificial fiber etc. *Sakiadis* [1, 2] was the pioneer in the work about the boundary layer but his work was not done for a closed form solutions. So, *Crane* [3] was the first to calculate the solution of steady and 2-D stretching surfaces in closed form in which the velocity is far from the boundary and is also proportional to distance from the fixed point.

Later, many other researchers like *Kumaran and Ramanaih* [5], *Miklavcic and Wang* [8], *Fang and Zhang* [13], *Fang et al.* [15], *Salleh et al.* [16] and *Qasim* [23] have discussed the layer about the boundary over the stretching/shrinking surfaces. The analysis of such medium around the surfaces of fluid flow on stretching/shrinking sheet has very significant feature that has not been dealt with properly till now. Though, there are some researchers who tempted to look into the study of flow in the stretching sheet and they just endeavored the law introduced by Henry Darcy who acknowledged the apparent equation developed for the movement of the flow in Newtonian fluid at a tiny Reynold's number. In accordance with this law, "The flow is linearly dependent upon the pressure gradient and the velocity produced is proportional to the pressure gradient."

Khan and Pop [17] and [21, 22] inspected the flows on boundary of normal nano-particles of fluids passing through a stretching/shrinking surfaces in various thermic conditions on the boundaries. They also utilized some of the models to evaluate the consequences of numbers with no dimensions and framework for the nano particles of the fluid on skin friction. There were many work discussed on shrinking sheet by some of the renowned researchers such as *Hayat et al.* [10], *Muhaimin et al.* [11] and *Wang* [12].

According to the French engineer Leveque in 1928, "The convective heat transfer in a flowing fluid is affected only by the velocity values very close to the surface." The flows affected from the very large Prandtl number may influence the changes in the mass/temperature beyond the thin area near the surface due to which the velocities in this area remain linear and have the distance normal to this surface. Here, heat transfer analysis is performed with many other models and their boundary conditions [20, 24, 26–29]

The recent research is attempted to analyze dual nature closed form solution for the given flow of the fluid and transfer of heat because of stretching/shrinking sheet inside medium through which the fluid can flow. That's why omission of the second order inertial term from the N-S equations are supposed to be done with the Brinkman-Forchheimer model [7]. Furthermore, the slow motion of flow has been considered so that the term of change in temperature vanishes in the momentum equations. The analysis in transfer of Heat has been executed for some models with non-identical boundary conditions. Consequently, the incomplete gamma function is obtained from the solutions of velocity and temperature profiles. The descriptions of these fields are then investigated precisely. The results obtained in this regard has been taken out in an excellent form.

The nanometer-sized particles in a fluid are called nanoparticles and such fluids are designed with the suspension of the mixture of nanoparticles in the base fluids like water, ethylene glycol and oil. All those particles with size less than 100 nanometers are of the characteristics varying from the usual solids. As compare to the micro-sized particles, nanoparticles have greater potential of the increase via heat transfer. Some researchers tempted hard to get highly affected heat transfer fluids from suspension of nanoparticles in the fluids. Choi (1995) remains the first to introduce the term nanofluids obtained from the suspension of the nanoparticles. The fundamental usage of the nanofluids are the increased thermal properties in heat transfer equipment as coolants and performance of heat transfer may be remarkably boosted by the suspension of the nanoparticles in heating or cooling fluids. Some of the values of physical properties like heat capacity, density and thermal conductivity have been used for both the base fluid (water) and nanofluid particle (Cu) [9].

Chapter 2

Fundamental concepts and definitions

Some of the fundamental definitions and few laws associated with the flow of fluid have been discussed in this chapter, briefly.

2.1 Fluid

The continuous deformation of a substance by applying stress on it, called fluid.

2.2 Fluid mechanics

The type of mechanics dealt with the properties of fluid in motion or at rest. It is subdivided into three categories. They are; fluid dynamics, kinematics and statics. Fluid dynamics deals with study of the motion of fluid's particles. Fluid kinematics deals with study of the motion of fluid's particles without any external force acting upon it and the fluid statics is the study of fluid's particles which are at rest.

2.3 Physical properties of fluid

2.3.1 Density

Density is the ratio of mass and volume of the fluid's particle and it is expressed by ρ. Mathematically,

$$\rho = \frac{m}{V}$$

The dimension of density is $[ML^{-3}]$.

2.3.2 Dynamic viscosity

The ratio of the shear stress and rate of deformation is called dynamic viscosity and is denoted by a Greek letter μ. Mathematically,

$$\mu = \frac{shear\ stress}{deformation\ rate}$$

Its dimension is $[L^2T^{-1}]$.

2.3.3 Kinematic viscosity

It is the change in dynamic viscosity of the fluid to density. It is expressed by ν. Mathematically kinematic viscosity is expressed as

$$\nu = \frac{dynamic\ viscosity}{density} = \frac{\mu}{\rho}$$

2.4 Classification of fluid

2.4.1 Inviscid fluid

The fluid with zero viscosity is called ideal fluid or inviscid fluid.

2.4.2 Real fluid

The fluid with viscosity which is not at zero, is called real fluid. This fluid is classified into two types, they are;

2.4.2.1 Newtonian fluid

It is the fluid that obeys Newton's law of viscosity. Some examples of such type of fluid are the air, water, mercury, etc. Mathematically,

$$\tilde{\tau}_{yx} = \mu \frac{du}{dy}$$

Here $\tilde{\tau}_{yx}$, u, $\frac{du}{dy}$ and μ are the shear stress, velocity component, deformation rate and viscosity, respectively.

2.4.2.2 Non-Newtonian fluid

It is such a fluid which does not obey Newton's law of viscosity. Blood, paints, ketchup, etc. are the examples. its mathematical expression is;

$$\tilde{\tau}_{yx} = k(\frac{du}{dy})^n$$

for $n \neq 1$, where, k and n are the consistency and behavior index, respectively.

2.4.3 Compressible fluid

If the density of fluid changes with temperature and pressure, then such kind of fluid is called a compressible fluid. One of most common example is of gases.

2.4.4 Incompressible fluid

If density is not depending upon the temperature and pressure, then such a fluid is known as incompressible fluid. In general, liquids are considered to be incompressible.

2.5 Two-Dimensional flow

Dimensions are basically the space coordinates and mostly the fluid motions are considered to be three dimensional but for the convenience in its calculation, it is taken to be two dimensional so that it can easily be dealt with. 2-D flow means flow to be in the plane coordinate.

2.6 Boundary layer

It is the fluid's layer near surface where viscosity of fluid is dominant. In general, boundary layer is the domain where the effects of the viscosity is dominant.

2.7 Porous medium

Medium containing a minute opening in the surface that allows passage of fluids through these holes are called porous medium. The objects consisting of the porous surfaces have void empty spaces or pores through which fluid particles penetrate the object. Sands, limestone, wooden materials, tissue papers, human lungs and biological tissues are some of examples existing in the nature and cements, foams and sponges are the man-made examples.

2.8 Heat and mass transfer

Kinetic process in which heat transfer is the movement of energy from one particle to another. Whereas mass transfer is the movement of mass from one place to another like absorption, evaporation etc. Hence, the efficiency of these two processes are considered collectively.

2.9 Nanofluids

The nano-meter sized particles in the suspension with some of the base fluids (water, oil, etc.) are called nanofluids. They exhibit enhanced thermal conductivity and the convective heat transfer coefficient compared to the base fluid. Nanofluids play an essential role in utilization of many applications in heat transfer equipment as coolants and radiators.

2.10 Thermal radiation

Thermal radiation is the generation of electromagnetic radiation with its thermal motion of particles in matter. The emission of thermal radiation of all matters is for the temperature greater than absolute zero.

2.11 Some useful non-dimensional numbers

2.11.1 Prandtl number

It is the non-dimensional number which is a change in kinematic viscosity ν with respect to thermal diffusivity α. Mathematically,

$$Pr = \frac{\nu}{\alpha}$$

It has no dimension.

2.11.2 Reynolds number

The non-dimensional number defining the change in the inertial forces to the viscous forces. Mathematically;

$$Re = \frac{ax^2}{\nu}$$

2.11.3 Biot number

It is a non-dimensional number defined as, When the heat transfer coefficient is being multiplied with the characteristic length and divided with thermal conductivity of the body. Generally, it can be expressed as;

$$Bi = \frac{L_c h}{k}$$

Here $L_c = \frac{Volume\ of\ body}{surface\ area}$, characteristic length, h and k are heat transfer coefficient and thermal conductivity, respectively.

2.11.4 Nusselt number

A dimensionless number which is the ratio between the convective and the conductive heat transfer at the boundary is called local Nusselt number. Mathematically, it is expressed as;

$$Nu_x = \frac{xh_x}{k}$$

Chapter 3

Dual nature exact solutions of viscous flow past a moving stretching/shrinking surface enclosed in a porous medium

The aim of this chapter is to investigate the flow of boundary layer and heat transfer of viscous fluid on a stretching/shrinking surface in porous medium with convective boundary conditions. Initially, we formulated governing equations of the flow. Later, with the application of similarity variables, we have transformed the governed nonlinear (PDEs) to the dimensionless nonlinear (ODEs) to obtain closed form solution of momentum and energy. The presence of some other parameters on energy equation can also be seen. This chapter is the review of [30].

3.1 Formulation of the problem

Let us Consider a flow to be steady, 2-D and incompressible on stretching/shrinking sheet in the porous medium. Assuming stretching velocity to be linear, that is, $\bar{u}_w(x) = bx$, where b is a positive constant. The consideration of the slow motion of flow vanishes the term of the change in temperature in momentum equations. The governing equations are;

$$\nabla \cdot V = 0 \tag{3.1}$$

$$\frac{\rho}{\epsilon}(V \cdot \nabla)V = div\tau - \frac{\epsilon\mu}{K}V \tag{3.2}$$

$$(\rho C_p)(V \cdot \nabla)T = \tau \cdot (\nabla V) - divq_c \tag{3.3}$$

Here, τ can be expressed as;

$$\tau = -pI + \mu_{eff} E_1 \tag{3.4}$$

Brinkman stated that μ_{eff} and μ are equal to each other but generally not equal [7], and E_1 is first Rivlin-Ericksen tensor, that is,

$$E_1 = (\nabla V) + (\nabla V)^T \tag{3.5}$$

$$\nabla V = gradV \tag{3.6}$$

$$q_c = -k\ (\nabla T) \tag{3.7}$$

For given problem, we define the velocity and temperature field as;

$$V = [\bar{u}(x,y), \bar{v}(x,y), 0] \quad \text{and} \quad T = T(x,y) \tag{3.8}$$

Using Eq. (3.8) in Eqs. (3.5) and (3.6), we get

$$\nabla V = \begin{pmatrix} \bar{u}_x & \bar{u}_y & 0 \\ \bar{v}_x & \bar{v}_y & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad (\nabla V)^T = \begin{pmatrix} \bar{u}_x & \bar{v}_x & 0 \\ \bar{u}_y & \bar{v}_y & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.9}$$

Now, by utilizing Eq. (3.9) in Eq. (3.5), we obtain

$$E_1 = \begin{pmatrix} 2\bar{u}_x & (\bar{u}_y + \bar{v}_x) & 0 \\ (\bar{v}_x + \bar{u}_y) & 2\bar{v}_y & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.10}$$

Substituting Eq. (3.10) in Eq. (3.4), it results

$$\tau = \begin{pmatrix} -p + 2\mu_{eff}(\bar{u}_x) & \mu_{eff}(\bar{u}_y + \bar{v}_x) & 0 \\ \mu_{eff}(\bar{v}_x + \bar{u}_y) & -p + 2\mu_{eff}\bar{v}_y & 0 \\ 0 & 0 & -p \end{pmatrix} \tag{3.11}$$

To express the matrix Eq. (3.11) in component form, we have

$$\tau_{xx} = -p+2\mu_{eff}(\bar{u}_x), \;\; \tau_{xy} = \tau_{yx} = \mu_{eff}(\bar{v}_x+\bar{u}_y) \tag{3.12}$$

$$\tau_{xz} = \tau_{zx} = \tau_{yz} = \tau_{zy} = 0, \quad \tau_{yy} = -p + 2\mu_{eff}(\bar{v}_y), \quad \tau_{zz} = -p \tag{3.13}$$

Using Eqs. (3.12) and (3.13) in Eq. (3.2), we get

$$\frac{\rho}{\epsilon}(\bar{u}\bar{u}_x + \bar{v}\bar{u}_y) = -\frac{\partial p}{\partial x} + \mu_{eff}\nabla^2\bar{u} - \frac{\epsilon\mu}{K}\bar{u} \tag{3.14}$$

$$\frac{\rho}{\epsilon}(\bar{u}\bar{v}_x + \bar{v}\bar{v}_y) = -\frac{\partial p}{\partial y} + \mu_{eff}\nabla^2\bar{v} - \frac{\epsilon\mu}{K}\bar{v} \tag{3.15}$$

$$0 = -\frac{\partial p}{\partial z} \tag{3.16}$$

In the R.H.S of Eq. (3.3), the first term is taken to be zero because of the absence of viscous dissipation and q_c from Eq. (3.7) gives;

$$\tau \cdot (\nabla V) = 0 \tag{3.17}$$

$$q_c = -k\ [T_x\ ,\ T_y\ ,\ 0] \tag{3.18}$$

Now, utilizing the above equations Eqs. (3.17) and (3.18) in Eq. (3.3), we obtain

$$\rho C_p\ (\bar{u}T_x + \bar{v}T_y) = k_m \nabla^2 T \tag{3.19}$$

The boundary conditions of above problem are

$$\bar{u} = \bar{u}_w(x) = ax, \quad \bar{v} = \bar{v}_w \ \text{ at } \ y = 0 \tag{3.20}$$

$$\bar{u}(y\ \rightarrow\ \infty)\ \rightarrow\ 0 \tag{3.21}$$

Here, $\bar{v}_w$, mass transfer on the surface for suction velocity ($\bar{v}_w < 0$) and injection velocity ($\bar{v}_w > 0$). Consequently, the conditions on the boundary of the convection appear as [14, 18, 19].

$$-k\ T_y|_{y=0} = \bar{h}_f(\bar{T}_f - T) \tag{3.22}$$

$$T(T\ \rightarrow\ \infty)\ \rightarrow\ \bar{T}_\infty \tag{3.23}$$

C_f and Nu_x can be written as

$$C_f = \frac{\bar{\tau}_w}{\rho \bar{u}_w^2}\ , \qquad Nu_x = \frac{x\bar{q}_w}{k_m(\bar{T}_w - T_\infty)} \tag{3.24}$$

$$\bar{\tau}_w = \mu(\bar{u}_y)\ , \quad \bar{q}_w = -k_m(T_y) \ \text{ at } \ y\ =\ 0 \tag{3.25}$$

We convert the given nonlinear PDEs to dimensionless nonlinear ODEs by introducing similarity variables [30] as

$$\bar{u} = axf'(\eta), \quad \bar{v} = -(a\nu)^{1/2}f(\eta), \quad \eta = y(\frac{a}{\nu})^{1/2}, \quad \theta(\eta) = \frac{T - \bar{T}_\infty}{\bar{T}_f - \bar{T}_\infty} \tag{3.26}$$

Here, prime symbolizes the differentiation of a function w.r.t. η. By applying the above similarity transformations Eq. (3.26) on Eqs. (3.14), (3.15) and (3.19), we obtain the dimensionless ODE as

$$\Xi f''' + ff'' - f'^2 - \Phi f' = 0 \tag{3.27}$$

$$\frac{1}{Pr}\theta'' + f\theta' = 0 \tag{3.28}$$

The reduced boundary conditions are

$$f(\eta = 0) = S \ , \ f'(\eta \to \infty) = 0 \ , \ f'(\eta = 0) = \frac{b}{a} = \alpha \tag{3.29}$$

$$\theta'(\eta = 0) = -Bi[1 + \theta(\eta = 0)] \ , \quad \theta(\eta \to \infty) = 0 \tag{3.30}$$

Where, S is the suction for ($S > 0$) or injection for ($S < 0$).

$$Pr = \frac{\nu}{\alpha}, \ \Xi = \epsilon\frac{\mu_{eff}}{\mu}, \ \Phi = \frac{\epsilon^2\nu}{aK}, \ Bi = \frac{h_f}{k_m}\sqrt{\frac{\nu}{a}}, \ \alpha = \frac{b}{a}, \ S = \frac{v_w}{\sqrt{b\nu}} \tag{3.31}$$

Also, using Eq. (3.26) in Eqs. (3.24) and (3.25), we get

$$(Re_x)^{1/2} C_f = f''(\eta = 0) \ , \quad Nu_x/(Re_x)^{1/2} = -\theta'(\eta = 0) \quad (3.32)$$

where, $Re_x = (ax^2/\nu)$, local Reynolds number.

To establish the solution of the transformed dimensionless nonlinear ODEs, we assume the solution of Eq. (3.27) satisfying boundary conditions as

$$f(\eta) = S + \frac{\alpha}{\beta}(1 - e^{-\beta\eta}) \quad (3.33)$$

Using Eq. (3.33) in Eq. (3.27), it yields

$$\Xi\beta^2 - S\beta - (\alpha + \Phi) = 0 \quad (3.34)$$

Solving the above equation for the value of β, we get

$$\beta = \frac{S \pm \sqrt{S^2 + 4\Xi(\alpha + \Phi)}}{2\Xi} \quad (3.35)$$

Eq. (3.35) shows the dual nature solution of the given problem. From Eqs. (3.25) and (3.32), we obtain

$$f'(\eta) = \alpha e^{-\beta\eta} \quad \text{and} \quad f''(0) = -\alpha\beta \quad (3.36)$$

To get the solution of dimensionless nonlinear ODE of energy equation, we consider a new variable ζ as follows

$$\zeta = \frac{Pr}{\beta^2} e^{-\beta\eta} \quad (3.37)$$

To apply this variable in Eq. (3.28), we convert the differentiation w.r.t η by using chain rule for first and second order ODEs, that is,

$$\frac{d}{d\eta} = \frac{d}{d\zeta} \cdot \frac{d\zeta}{d\eta} \quad \text{and} \quad \frac{d^2}{d\zeta^2}(\frac{d\zeta}{d\eta})^2 + \frac{d}{d\zeta} \cdot \frac{d^2\zeta}{d\eta^2} \tag{3.38}$$

After applying the above chain rule on Eq. (3.28), we obtian

$$\zeta\frac{d^2\theta}{d\zeta^2} + (1 - Pr + \alpha\zeta)\frac{d\theta}{d\zeta} = 0 \tag{3.39}$$

and the reduced boundary conditions are

$$\frac{Pr}{\beta}\ \theta'(\frac{Pr}{\beta^2}) = Bi\ [1 - \theta(\frac{Pr}{\beta^2})] \quad , \quad \theta(0) = 0 \tag{3.40}$$

Considering $\frac{d\theta}{d\zeta} = \omega$ and using in Eq. (3.39), we get

$$\frac{d\omega}{d\zeta} + (\frac{1 - Pr}{\zeta} + \alpha)\omega = 0 \tag{3.41}$$

$$\implies \quad \frac{d}{d\zeta}(\zeta^{1-Pr}e^{\alpha\zeta}\omega) = 0 \tag{3.42}$$

On integrating Eq. (3.42), it gives

$$\zeta^{1-Pr}e^{\alpha\zeta}\omega = C_1 \tag{3.43}$$

or

$$\omega = C_1\zeta^{Pr-1}e^{-\alpha\zeta} \tag{3.44}$$

Using $\omega = \frac{d\theta}{d\zeta}$ in Eq. (3.44) and integrating, we get

$$\theta(\zeta) = C_1 \int_0^\infty \zeta^{Pr-1}\, e^{-\alpha\zeta}\, d\zeta \tag{3.45}$$

$$\implies \quad \theta(\zeta) = C_1 \Gamma(Pr, \alpha\zeta) + C_2 \tag{3.46}$$

where, $\Gamma(a,x) = \int_0^\infty t^{a-1} e^{-t} dt$ is the incomplete Gamma function. C_1 and C_2 are the constants of integration. Using ζ in Eq. (3.46), we get

$$\theta(\eta) = C_1 \Gamma(Pr, \frac{\alpha Pr}{\beta^2} e^{-\beta\eta}) + C_2 \tag{3.47}$$

Differentiating w.r.t η, we get

$$\theta'(\eta) = C_1 \beta e^{-\frac{\alpha Pr}{\beta^2} e^{-\beta\eta}} (\frac{\alpha Pr}{\beta^2} e^{-\beta\eta})^{Pr} \tag{3.48}$$

Applying boundary condition Eq. (3.30) in Eqs. (3.47) and (3.48), it gives

$$C_1 = \frac{-Bi}{\beta e^{-\frac{\alpha Pr}{\beta^2}} (\frac{\alpha Pr}{\beta^2})^{Pr} + Bi\ \Gamma(Pr, \frac{\alpha Pr}{\beta^2}) - Bi\ \Gamma(Pr, 0)} \tag{3.49}$$

$$C_2 = \frac{Bi\ \Gamma(Pr, 0)}{\beta e^{-\frac{\alpha Pr}{\beta^2}} (\frac{\alpha Pr}{\beta^2})^{Pr} + Bi\ \Gamma(Pr, \frac{\alpha Pr}{\beta^2}) - Bi\ \Gamma(Pr, 0)} \tag{3.50}$$

Utilizing C_1 and C_2 from Eqs. (3.49) and (3.50) in Eq. (3.47), we obtain the solution as,

$$\theta(\eta) = \frac{Bi\ \Gamma(Pr, 0) - Bi\ \Gamma(Pr, \frac{\alpha Pr}{\beta^2} e^{-\beta\eta})}{\beta e^{-\frac{\alpha Pr}{\beta^2}} (\frac{\alpha Pr}{\beta^2})^{Pr} + Bi\ \Gamma(Pr, \frac{\alpha Pr}{\beta^2}) - Bi\ \Gamma(Pr, 0)} \tag{3.51}$$

3.2 Results and discussion

In order to narrate variations of dimensionless ODE of momentum, energy, skin friction and Nusselt number with some parameters. Figs. (1) - (3) determine different parameters on the domain of solution β, in this situation, amazing behavior is observed. Physically, the change occurred in upper and lower branches of the solution for some specific values assigned to the S, Ξ and α. Here, the $(+)$ sign in the solution Eq. (3.35) corresponds to upper branch and $(-)$ sign to lower branch. For the fixed values of Ξ, Φ and $-\alpha$, β increases for the increasing value of S (for upper branch) else reduces for lower branch. Figs. (4) - (7) show changes in the skin friction coefficient by using different physical parameters. Increase in the value of α (shrinking case only) decreases the skin friction coefficient whereas it can also be seen that the increase in Φ and Ξ increased C_f for both branches of solution. Larger C_f only exists for suction rather than injection.

Figs. (8) and (9) illustrate the variations of S and α on non-dimensional velocity for both branches of dual nature solution. S increases the dimensionless velocity in lower branch which causes an increase in boundary layer thickness and increase in skin friction takes place whereas the upper branch illustrates opposite behavior.In Fig. (9), effects of α on dimensionless velocity is examined in both the branches where it is found that Eq. (3.29) obviously shows the dependence of stretching parameter on the non-dimensional velocity profile. Increasing α causes decrease in the velocity profile on upper branch which consequences decrease in skin friction. Figs. (10) and (11) portrayed the variations in Phi and Ξ on non-dimensional velocity, respectively. These effects are opposite on both the parameters inside the boundary layer thickness in upper and lower branches. Fig. (10) shows increase in resistance of flow by increasing Ξ in upper branch however, in Fig. (11), Φ makes no eminent effect on non-dimensional velocity in upper branch.

In Figs. (12a) and (12b), streamlines are plotted for the value of stretching parameter at $\alpha = -0.05$, Fig. (13a) and (13b) shows the streamlines at $\alpha = -0.1$, and Fig. (14a) and (14b) depict the streamlines at $\alpha = -0.3$ which express the effects of the patterns of flow. The patterns in upper branch are quite different than that of the lower branch. Figs. (15) and (16) illustrate the change in Nusselt number with S and Φ in which Nusselt number increases with an increase in Biot number on both branches of solution. In Figs. (17) and (18), the effects of S and Bi are depicted on non-dimensional temperature profile. The result is evident that increasing S, decreases non-dimensional temperature due to thermal boundary layer. Consequently, the convective boundary layer thickness decreases dual solution and no-dimensional temperature is dependent on convective heat transfer coefficient due to which it is derived that heat transfer will be greater for its greater coefficient.

Figs. (19) and (20) show the variations of the dimensionless temperature due to Ξ and Φ on both branches. In Fig. (19), Ξ is increased by the dimensionless temperature in the presence of suction parameter S due to which thermal boundary layer thickness also increased. Whereas, Fig. (20) depicts the effects of Φ that reduces non-dimensional temperature in the thermal boundary layer for the upper and lower branch solutions.

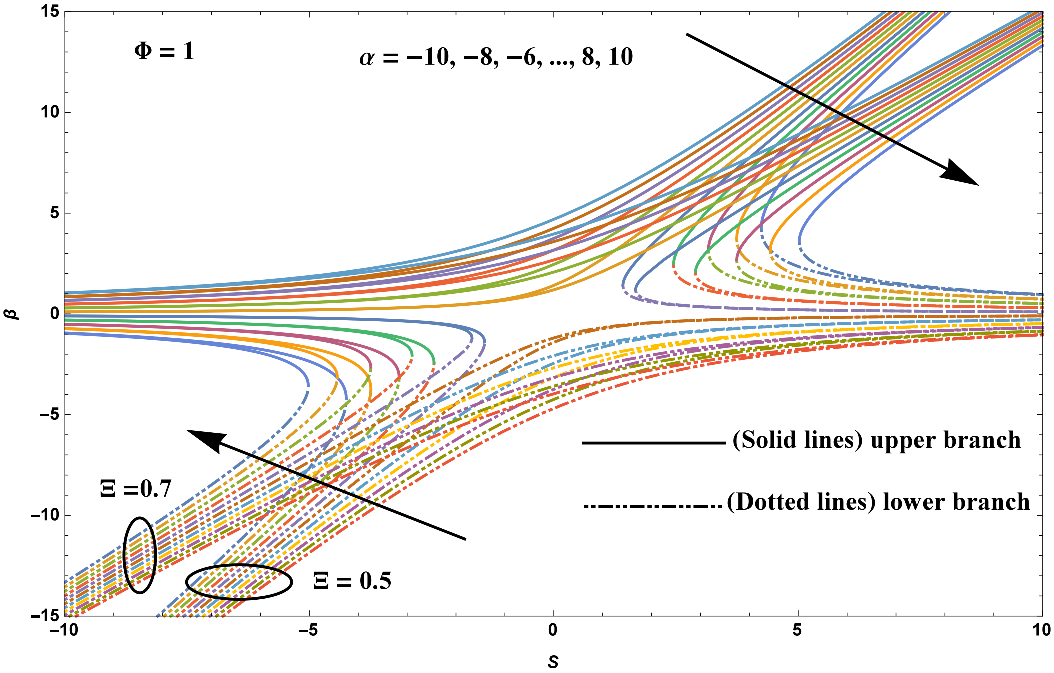


Fig. 1: Solution β as function of S

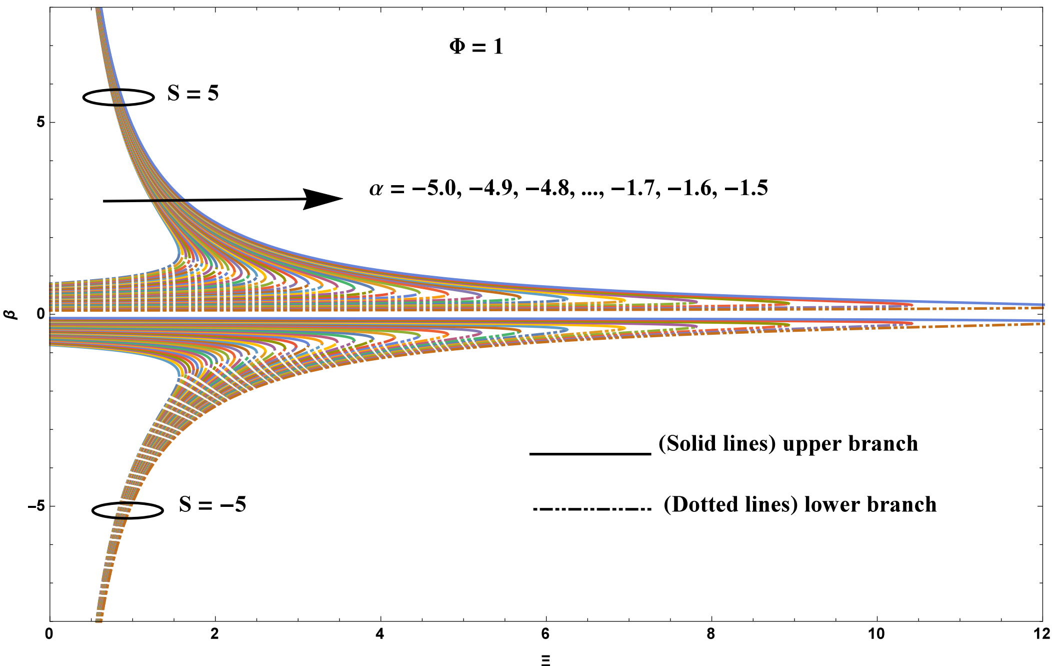


Fig. 2: Solution β as function of Ξ

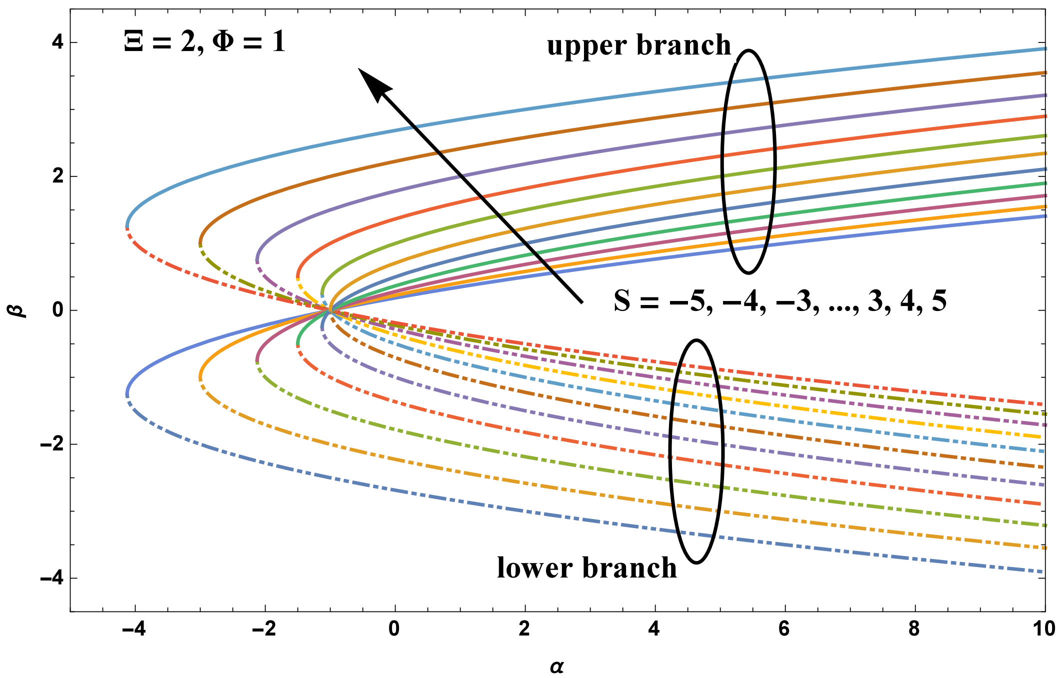


Fig. 3: Solution β as function of α

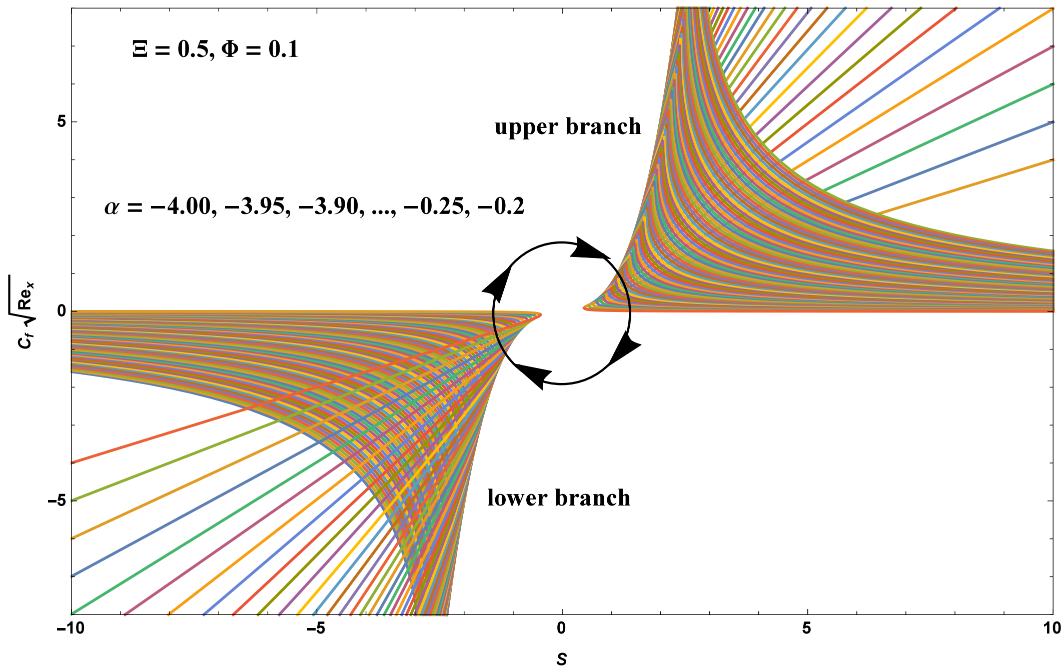


Fig. 4: Variations of skin friction with S

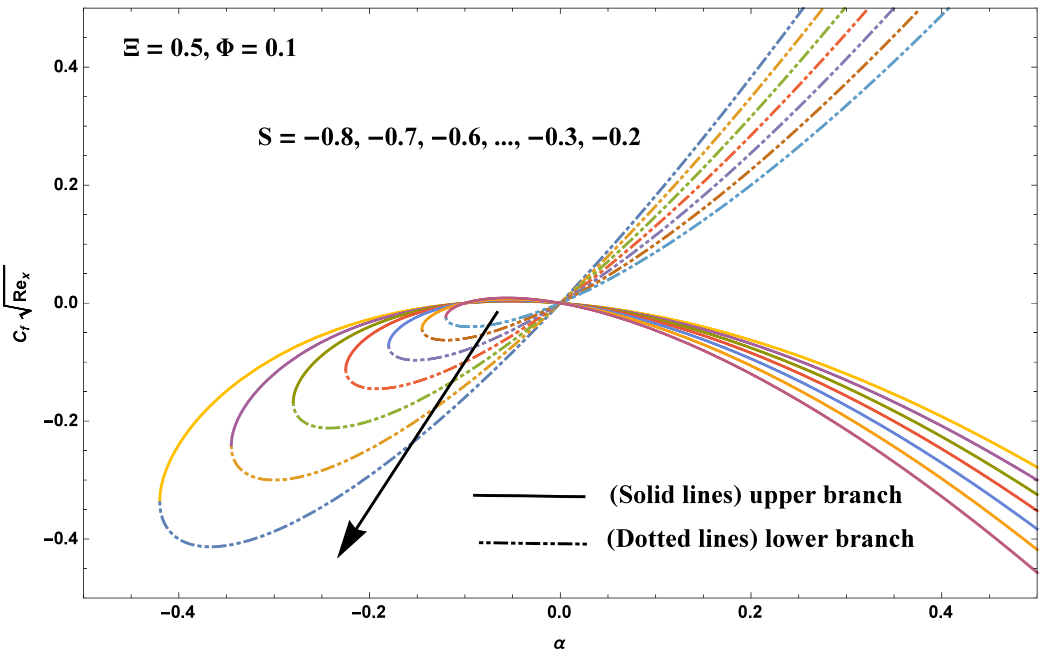


Fig. 5: Variations of skin friction with α

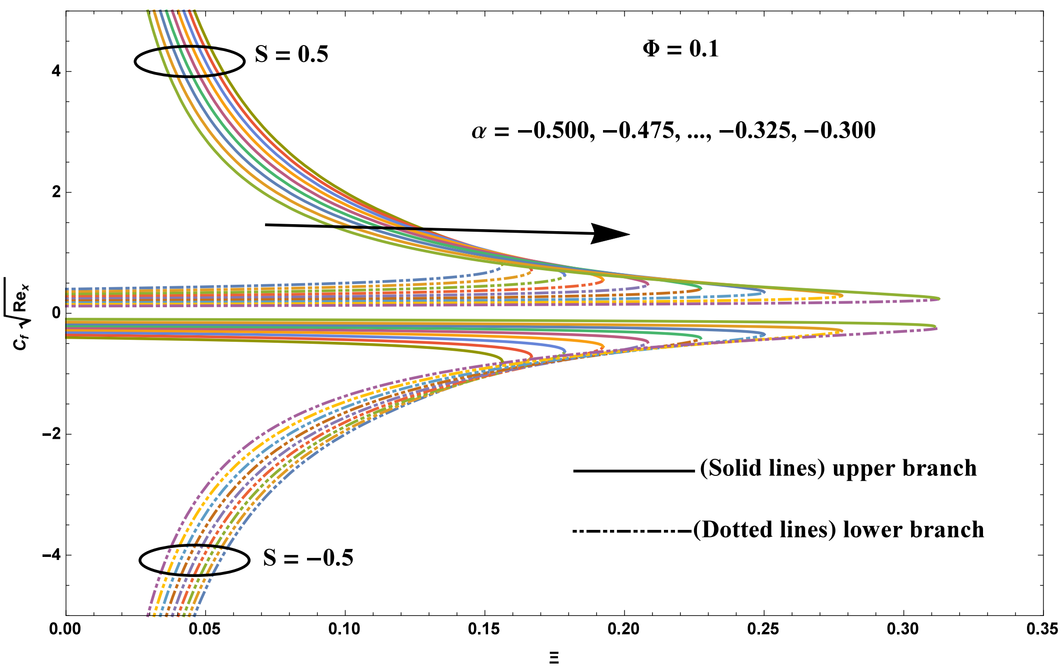


Fig. 6: Variations of skin friction with Ξ

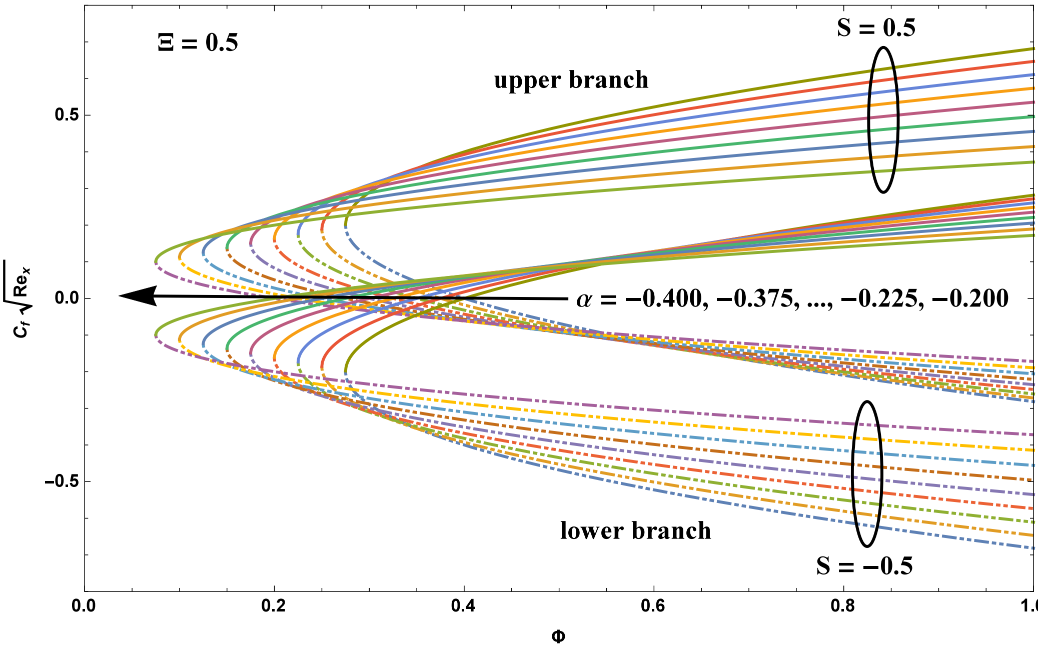


Fig. 7: Variations of skin friction with Φ

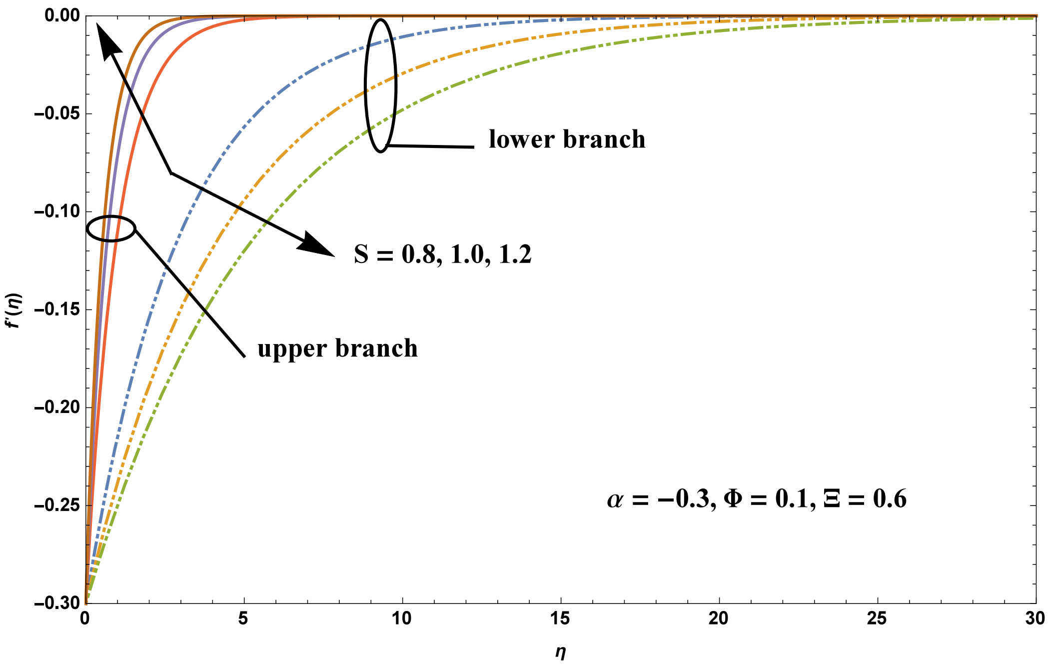


Fig. 8: Effects of suction parameter on velocity profile

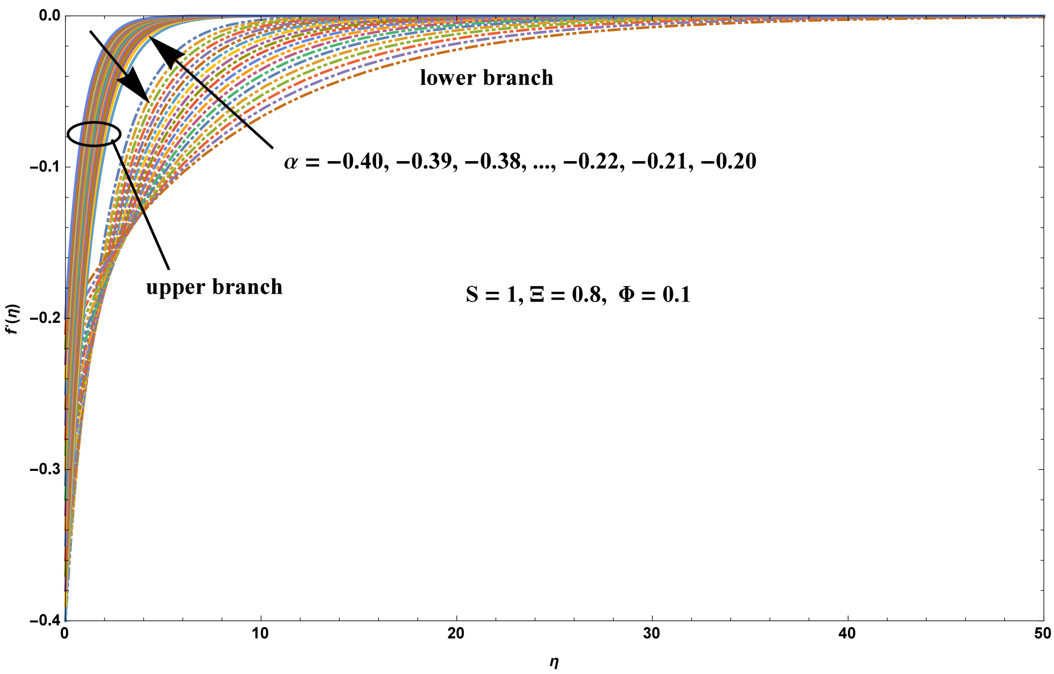


Fig. 9: Effects of stretching parameter on velocity profile

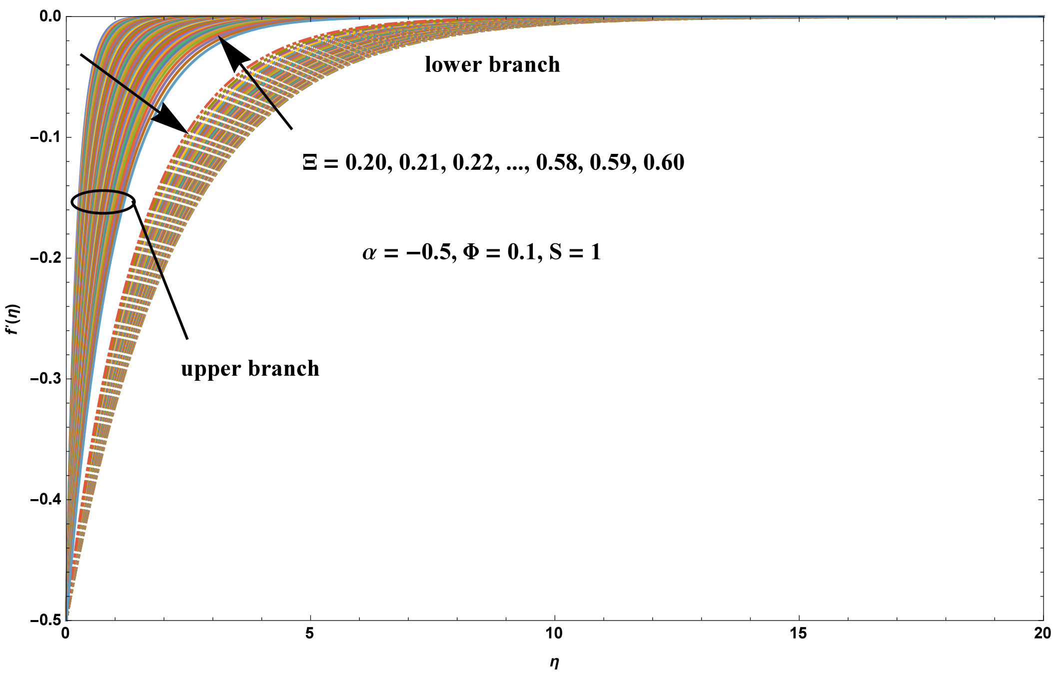


Fig. 10: Effects of viscosity ratio parameter on velocity profile

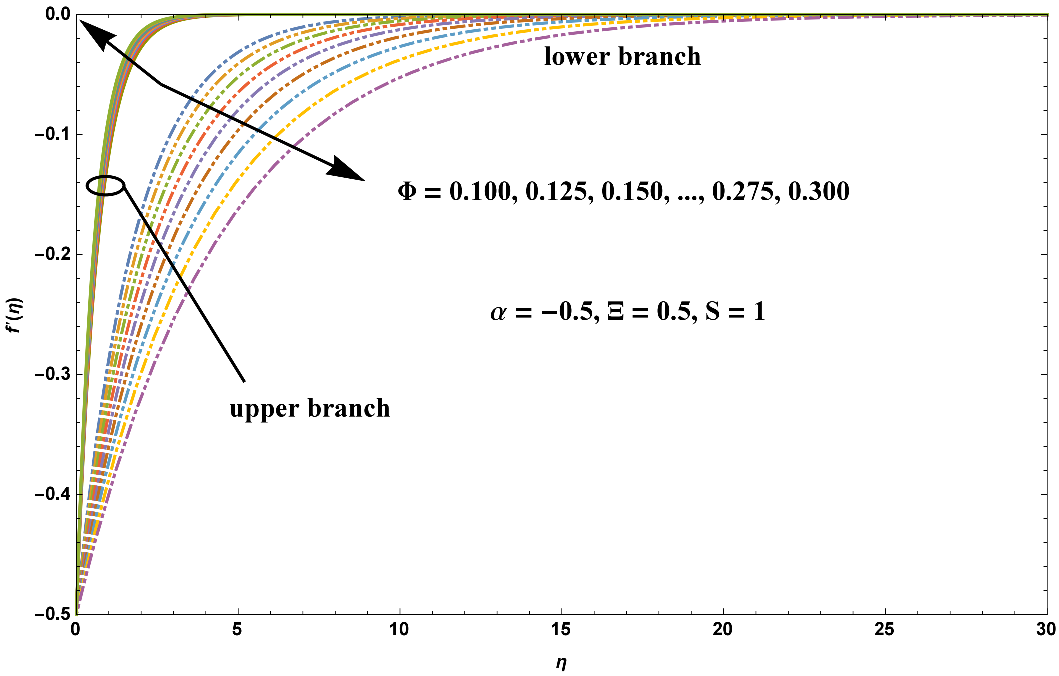
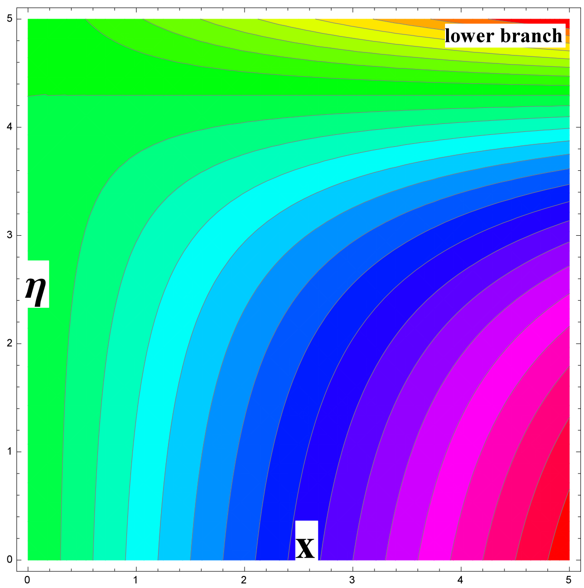
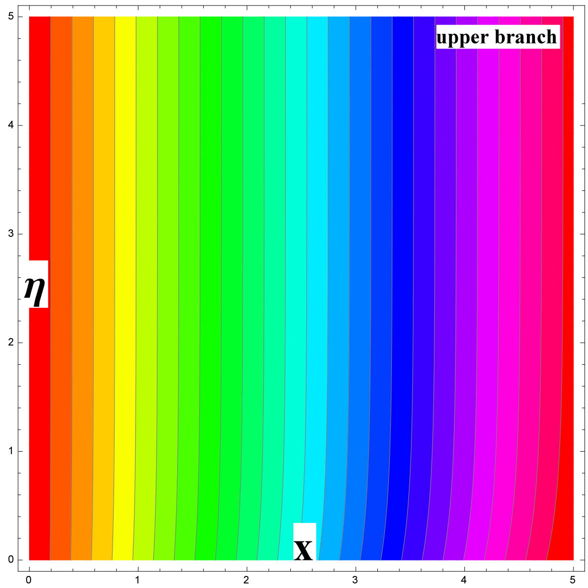


Fig. 11: Effects of porous medium parameter on velocity profile

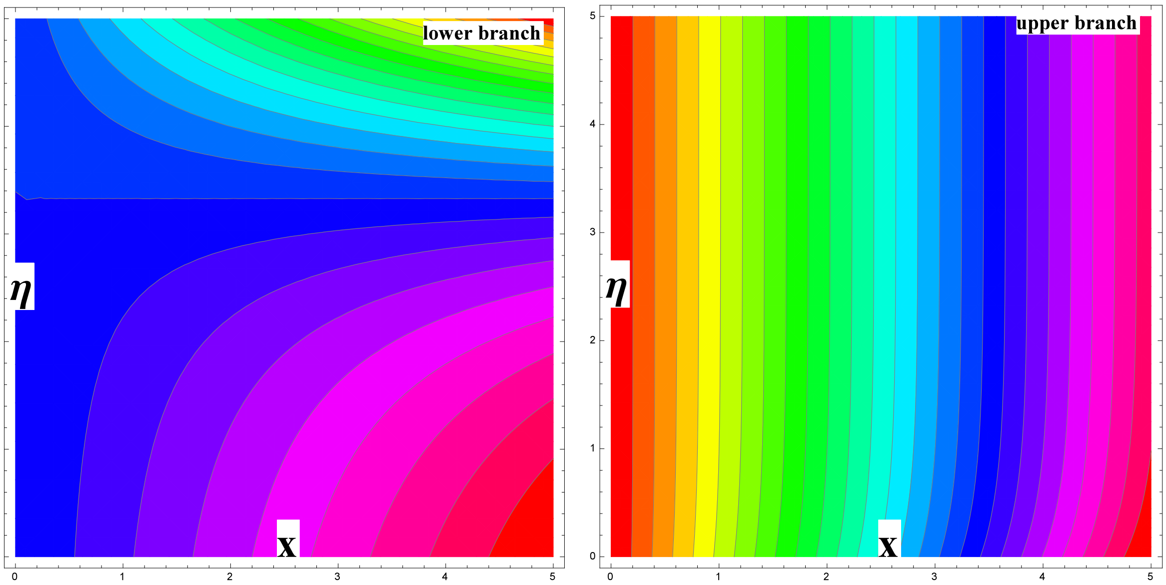


(a) Streamlines at $\alpha = -0.05$



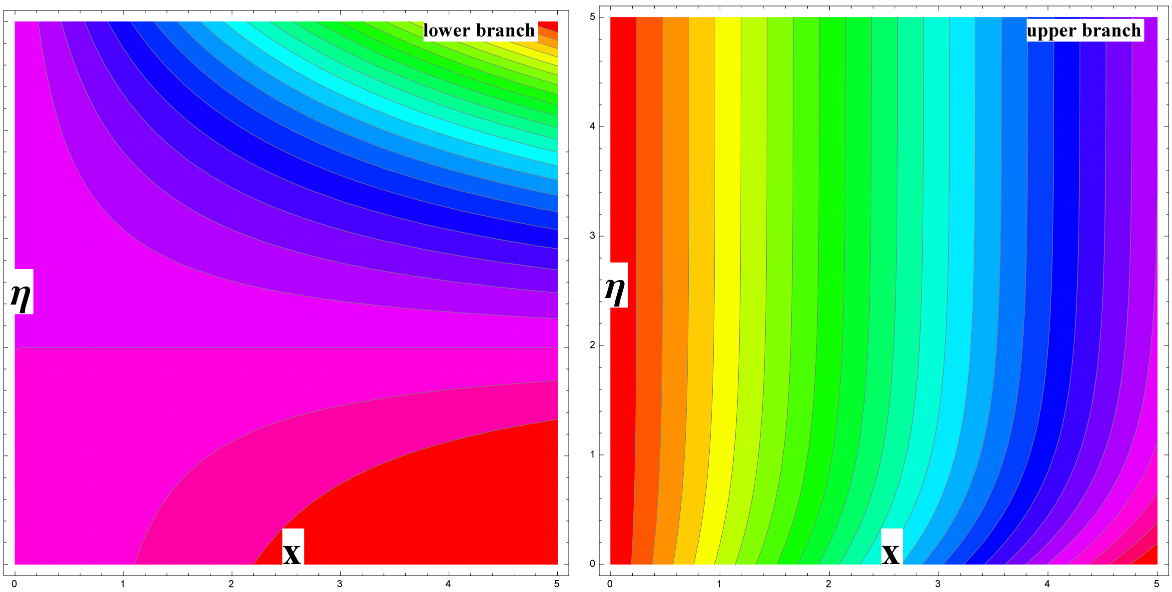
(b) Streamlines at $\alpha = -0.05$

Fig. 12: Streamlines for stretching parameter, $\alpha = -0.05$



(a) Streamlines at $\alpha = -0.1$ (b) Streamlines at $\alpha = -0.1$

Fig. 13: Streamlines for stretching parameter, $\alpha = -0.1$



(a) Streamlines at $\alpha = -0.3$ (b) Streamlines at $\alpha = -0.3$

Fig. 14: Streamlines for stretching parameter, $\alpha = -0.3$

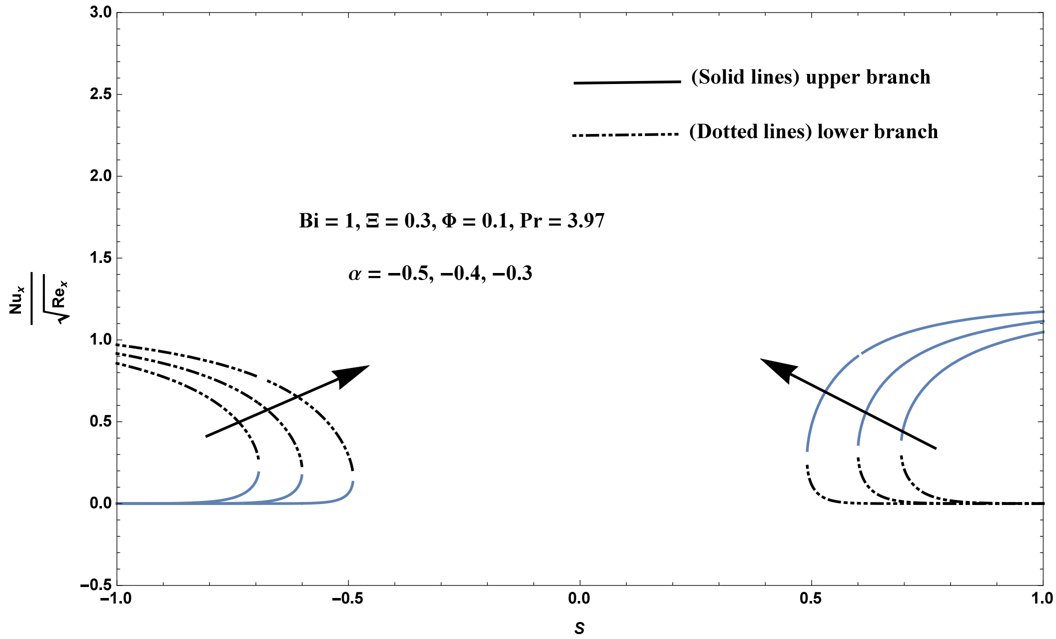


Fig. 15: Variations of Nusselt number with suction parameter

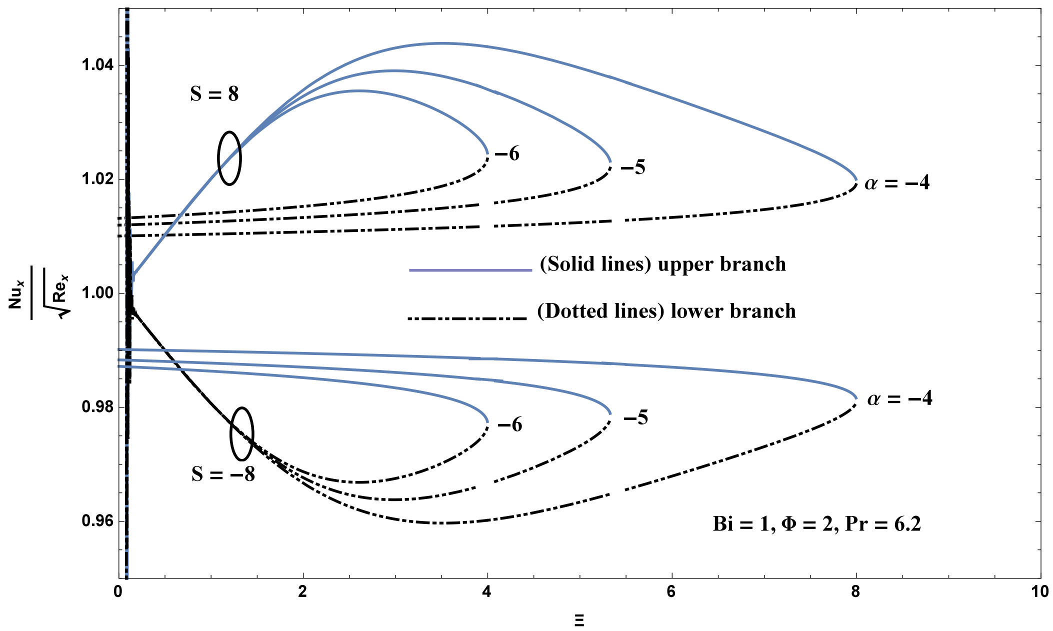


Fig. 16: Variations of Nusselt number with viscosity ratio parameter

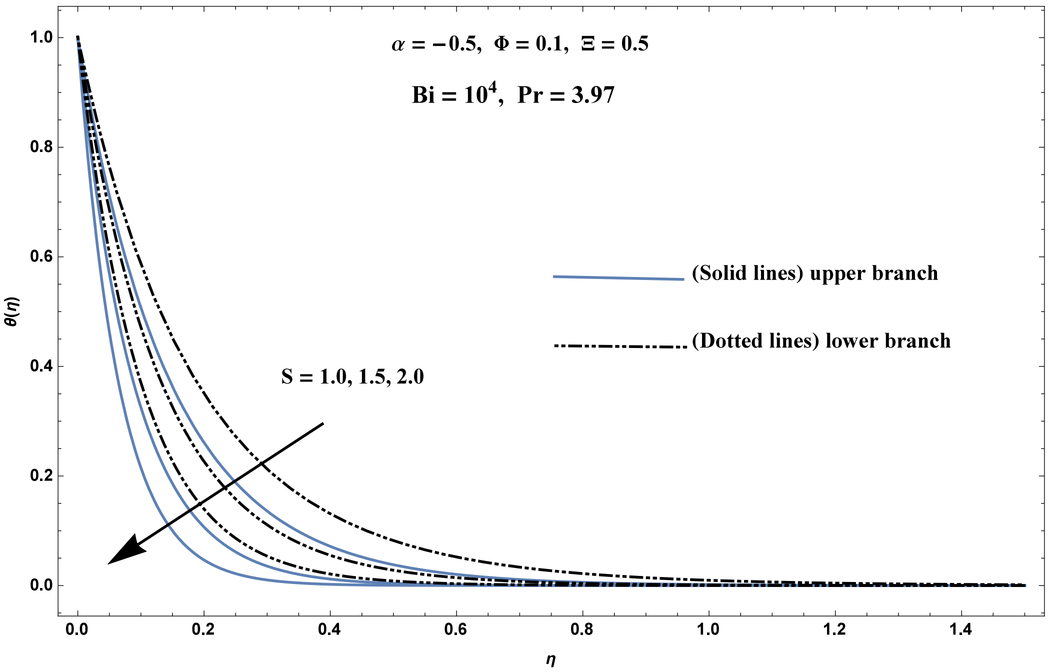


Fig. 17: Effects of suction parameter on temperature profile

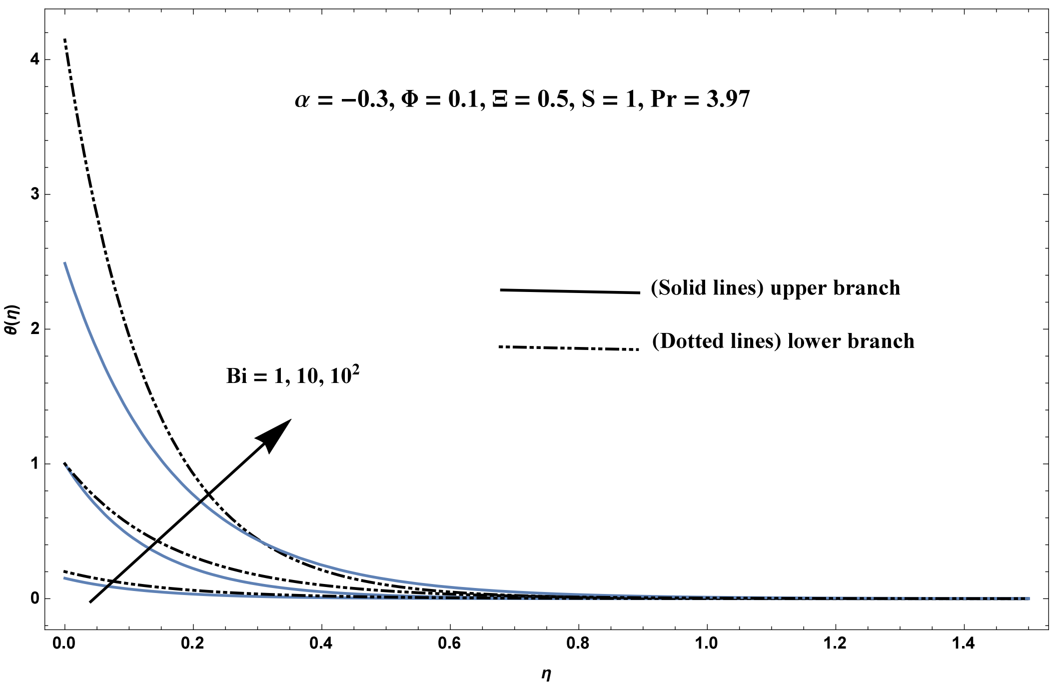


Fig. 18: Effects of Biot number on temperature profile

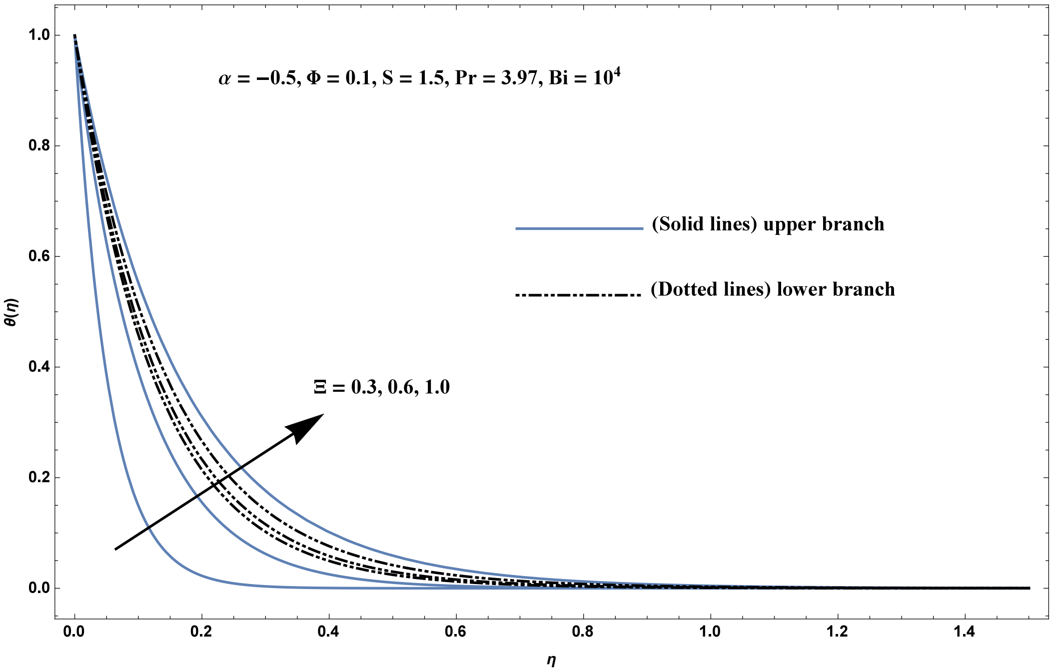


Fig. 19: Effects of viscosity ratio parameter on temperature profile

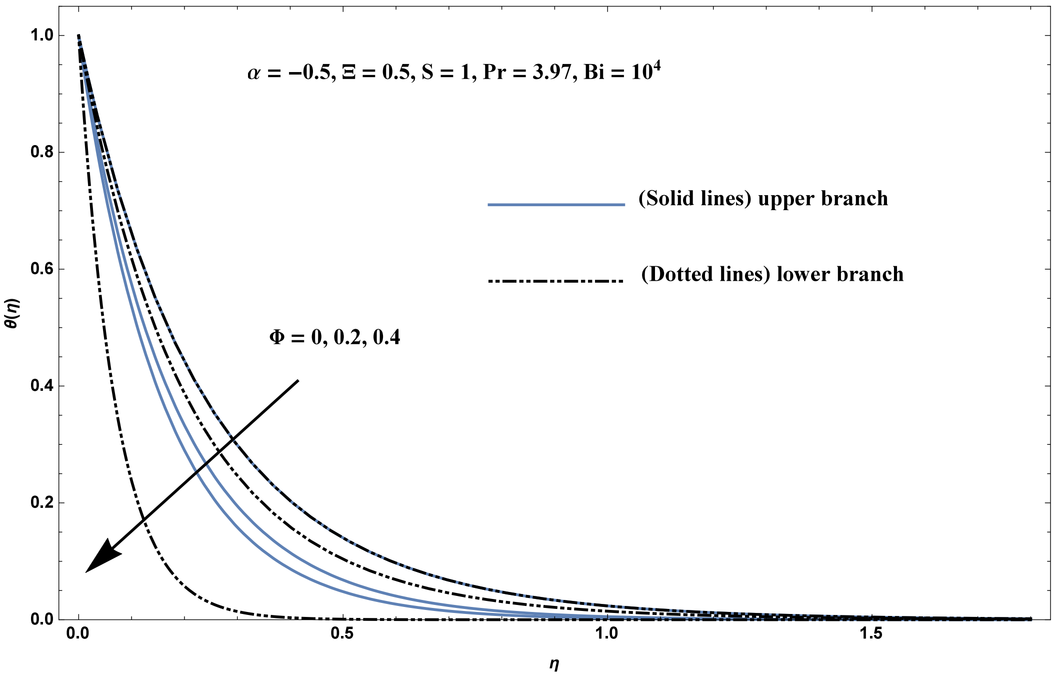


Fig. 20: Effects of porous medium parameter on temperature profile

Chapter 4

Dual nature study of convective heat transfers of nanofluid along a shrinking surface in a porous medium with thermal radiation

This chapter is the extension of previous chapter. Here the effect of nanofluid particle (Cu) and thermal radiation is studied with all physical properties graphically. The thorough study of heat transfers of nano-particles of fluid along a shrinking surface in porous medium with thermal radiation have been discussed. Later, the mathematical formulation modeled accordingly with their boundary conditions. For a mathematical formaulation, all the PDEs of momentum and energy are transformed into nonlinear ODEs by using the similarity variables. Lastly, the nonlinear ODEs are computed by incomplete gamma function to get dual nature closed form solutions.

4.1 Mathematical modelling and exact solution

Let us consider the steady flow across the boundary passing through a shrinking sheet in porous medium with linear velocity $\bar{u}_w(x) = bx$, where b is positive constant. The governing equations are;

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0 \tag{4.1}$$

$$\frac{\rho_{nf}}{\epsilon}(V \cdot \nabla)V = div\tau - \frac{\epsilon \mu_{nf}}{K}V \tag{4.2}$$

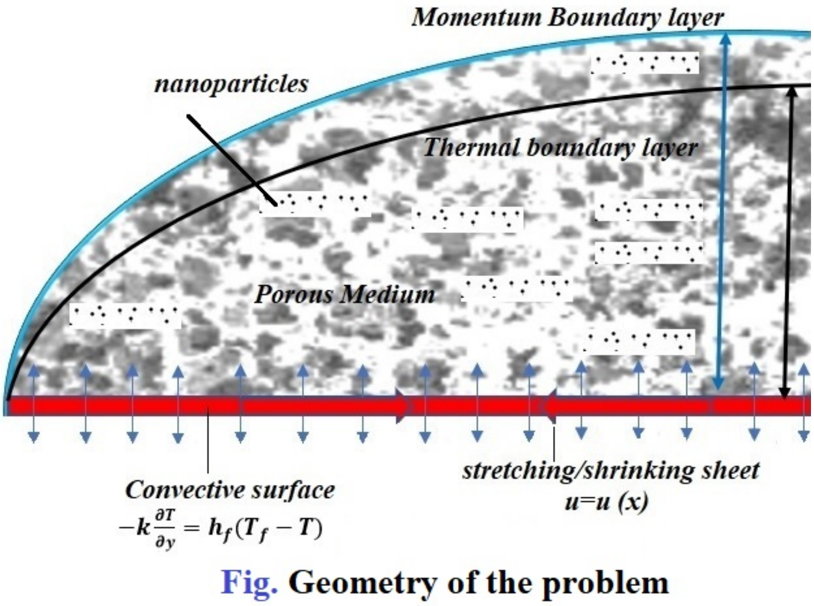


Fig. 21: Geometry of the problem

$$(\rho C_p)_{nf}(V \cdot \nabla)T = \kappa_{nf}(\nabla^2 T) - \frac{\partial q_r}{\partial y} \tag{4.3}$$

Now, τ for Newtonian fluid is expressed as:

$$\tau = -pI + \mu_{eff} E_1 \tag{4.4}$$

Where, $E_1 = (gradV) + (gradV)^T$, is the first Rivlin-Erickson tensor, μ_{eff} is the effective velocity and according to Brinkman, effective and dynamic viscosities are equal particularly but not in general [7] and ϵ is the dimensionless Forchheimer drag constant. Using Rosseland approximation [4]

$$q_r = -(\frac{4\sigma}{3K})(\frac{\partial T^4}{\partial y}) \tag{4.5}$$

Here, $T^4 = 4\bar{T}_\infty^3 T - 3\bar{T}_\infty^4$, using this value in Eq.(4.5), we get

$$q_r = -(\frac{16\sigma\bar{T}_\infty^3}{3K})(\frac{\partial T}{\partial y}) \tag{4.6}$$

Eq. (4.3) gives

$$(\rho C_p)_{nf}(V\cdot\nabla)T = \kappa_{nf}(\nabla^2 T) + (\frac{16\sigma\bar{T}_\infty^3}{3K})(\frac{\partial T}{\partial y}) \tag{4.7}$$

Assuming the fields for velocity and temperature as;

$$V = [\bar{u}(x,y), \bar{v}(x,y), 0] \quad \text{and} \quad T = T(x,y) \tag{4.8}$$

Now, utilizing velocity field from Eq. (4.8) in Eqs. (4.1 − 4.2), Continuity equation (4.1) holds and nonlinear partial differential equation of momentum in component form yields

$$\frac{\rho_{nf}}{\epsilon}(\bar{u}\frac{\partial\bar{u}}{\partial x} + \bar{v}\frac{\partial\bar{u}}{\partial y}) = (div\tau)_x - \frac{\epsilon\mu_{nf}}{K}\bar{u} \tag{4.9}$$

$$\frac{\rho_{nf}}{\epsilon}(\bar{u}\frac{\partial\bar{v}}{\partial x} + \bar{v}\frac{\partial\bar{v}}{\partial y}) = (div\tau)_y - \frac{\epsilon\mu_{nf}}{K}\bar{v} \tag{4.10}$$

$$0 = (div\tau)_z \tag{4.11}$$

After the calculation of first Rivlin-Erickson tensor using velocity field, we obtain

$$E_1 = \begin{pmatrix} 2\bar{u}_x & \bar{u}_y + \bar{v}_x & 0 \\ \bar{v}_x + \bar{u}_y & 2\bar{v}_y & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.12}$$

Substituting Eq. (4.12) in Eq. (4.4), it gives

$$\tau = \begin{pmatrix} -p + 2\mu_{eff}\frac{\partial \bar{u}}{\partial x} & \mu_{eff}(\frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x}) & 0 \\ \mu_{eff}(\frac{\partial \bar{v}}{\partial x} + \frac{\partial \bar{u}}{\partial y}) & -p + 2\mu_{eff}\frac{\partial \bar{v}}{\partial y} & 0 \\ 0 & 0 & -p \end{pmatrix} \tag{4.13}$$

τ in component form is expressed as

$$\tau_{xx} = -p+2\mu_{eff}\frac{\partial \bar{u}}{\partial x},\ \tau_{xy} = \tau_{yx} = \mu_{eff}(\frac{\partial \bar{v}}{\partial x}+\frac{\partial \bar{u}}{\partial y}) \tag{4.14}$$

$$\tau_{xz} = \tau_{zx} = \tau_{yz} = \tau_{zy} = 0, \quad \tau_{yy} = -p+2\mu_{eff}\frac{\partial \bar{v}}{\partial y}, \quad \tau_{zz} = -p \tag{4.15}$$

Utilizing Eqs. (4.14) and (4.15) in Eqs. (4.9) – (4.11), we get

$$\frac{\rho_{nf}}{\epsilon}(\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial y}) = -\frac{\partial p}{\partial x}+\mu_{eff}(\frac{\partial^2 \bar{u}}{\partial x^2}+\frac{\partial^2 \bar{u}}{\partial y^2})-\frac{\epsilon\mu_{nf}}{K}\bar{u} \tag{4.16}$$

$$\frac{\rho_{nf}}{\epsilon}(\bar{u}\frac{\partial \bar{v}}{\partial x}+\bar{v}\frac{\partial \bar{v}}{\partial y}) = -\frac{\partial p}{\partial y}+\mu_{eff}(\frac{\partial^2 \bar{v}}{\partial x^2}+\frac{\partial^2 \bar{v}}{\partial y^2})-\frac{\epsilon\mu_{nf}}{K}\bar{v} \tag{4.17}$$

$$0 = -\frac{\partial p}{\partial z} \tag{4.18}$$

The nonlinear PDE of energy equation (4.7) after the application of the temperature field for $\tau \cdot (\nabla V) = 0$, is expressed as

$$(\rho C_p)_{nf}(\bar{u}\frac{\partial T}{\partial x} + \bar{v}\frac{\partial T}{\partial y}) = \kappa_{nf}(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}) + (\frac{16\sigma\bar{T}_\infty^3}{3K})(\frac{\partial T}{\partial y}) \tag{4.19}$$

The boundary conditions for the momentum and energy equations are

$$\bar{u} = \bar{u}_w(x) = bx,\ \ \bar{v} = \bar{v}_w \quad \text{at} \quad y = 0 \tag{4.20a}$$

$$\bar{u}(y\ \rightarrow\ \infty)\ \rightarrow\ 0 \tag{4.20b}$$

$$-k\,(\tfrac{\partial T}{\partial y})|_{y=0} = \bar{h}_f(\bar{T}_f - T) \quad \text{and} \quad T(T\ \rightarrow\ \infty)\ \rightarrow\ \bar{T}_\infty \tag{4.20c}$$

Assuming the following similarity transformation to reduce the governing PDEs into the dimensionless nonlinear ODEs

$$\bar{u} = axf'(\eta),\ \ \bar{v} = -(a\nu)^{1/2}f(\eta),\ \ \eta = y(\frac{a}{\nu})^{1/2},\ \ \theta(\eta) = \frac{T - \bar{T}_\infty}{\bar{T}_f - \bar{T}_\infty} \tag{4.21}$$

$$(\rho C_p)_{nf} = (1-\phi)(\rho C_p)_f + \phi(\rho C_p)_s,\ \ \mu_{nf} = \frac{\mu_f}{(1-\phi)^{2.5}} \tag{4.22a}$$

$$\rho_{nf} = (1-\phi)\rho_f + \phi\rho_s,\ \ \frac{\nu_{nf}}{\nu_f} = \frac{1}{(1-\phi)^{2.5}[(1-\phi) + \phi\frac{\rho_s}{\rho_f}]} \tag{4.22b}$$

$$\frac{k_{nf}}{k_f} = \frac{(k_s + 2k_f) - 2\phi(k_f - k_s)}{(k_s + 2k_f) + \phi(k_f - k_s)} \tag{4.22c}$$

Applying above Eqs. (4.21) and (4.22(a, b, c)) in Eqs. (4.16) – 4.19), we get

$$\Xi\ A_1\ f''' + ff'' - f'^2 - \Phi\ A_2\ f' = 0 \tag{4.23}$$

$$\frac{1}{Pr}(1 + \frac{4}{3}R)\ A_3\ \theta'' + f\theta' = 0 \tag{4.24}$$

$$f(0) = S,\ \ f'(0) = b/a = \alpha,\ \ f'(\infty) = 0 \tag{4.25}$$

$$\theta'(\eta = 0) = -Bi[1+\theta(\eta = 0)], \quad \theta(\eta \to \infty) \to 0 \tag{4.26}$$

Here, the physical parameters are defined as

$$Pr = \frac{\nu_f}{\alpha_f}, \quad \Xi = \epsilon\frac{\mu_{eff}}{\mu_f}, \quad \Phi = \frac{\epsilon^2\nu_f}{bK}, \quad S = \frac{v_w}{\sqrt{b\nu}}$$

$$Bi = \frac{h_f}{k_m}\sqrt{\frac{\nu_f}{b}}, \quad A_1 = \frac{\nu_{nf}}{\nu_f}, \quad A_2 = A_1(1-\phi)^{2.5}$$

$$A_3 = \frac{(k_s + 2k_f) - 2(k_f - k_s)\phi}{((k_s + 2k_f) + \phi(k_f - k_s))((1-\phi) + \phi\frac{(\rho C_p)_s}{(\rho C_p)_f})} \tag{4.27}$$

Where ν_f *and* ν_{nf} are kinematic viscosity of fluid and nanofluid parameters, respectively. C_f and Nu_x are expressed below;

$$C_f = \frac{\bar{\tau}_w}{\rho \bar{u}_w^2}, \quad Nu_x = \frac{x\bar{q}_w}{k_m(\bar{T}_w - \bar{T}_\infty)} \tag{4.28}$$

Here the skin friction at wall and the heat transfer from the plate can be presented as

$$\bar{\tau}_w = \mu_{nf}(\bar{u}_y)|_{y=0}, \quad \bar{q}_w = -k_{nf}(T_y)|_{y=0} \tag{4.29}$$

After applying similarity transformation Eq. (4.21) in Eqs. (4.28) and (4.29), we obtain [25]

$$(Re_x)^{1/2}C_f = \frac{\mu_{nf}}{\mu_f}f''(0), \quad Nu/Re_x^{1/2} = -\frac{k_{nf}}{k_f}\theta'(0) \tag{4.30}$$

Here $Re_x = (ax^2/\nu)$, the local Reynolds number.

4.2 Methodology

The dual nature solution of Eq. (4.23) satisfying the constraints in Eq. (4.25) is assumed as

$$f(\eta) = S + \frac{\alpha}{\beta}(1 - e^{-\beta\eta}) \tag{4.31}$$

Using the above equation in Eq. (4.23), the β solution yields

$$\beta = \frac{S \pm \sqrt{S^2 + 4A_1\Xi(\alpha + A_2\Phi)}}{2A_1\Xi} \tag{4.32}$$

It is evident from Eq. (4.32) that the proposed problem has dual solution. Eqs. (4.29) and (4.30) depict $f'(\eta)$ and $f''(0)$ at the surface, they are

$$f'(\eta) = \alpha e^{-\beta\eta}, \; and \; f''(0) = -\alpha\beta \tag{4.33}$$

Introducing a new variable, ξ, to get the solution of dimensionless nonlinear ODE of energy equation, that is,

$$\xi = \frac{Pr}{\beta^2} e^{-\beta\eta} \tag{4.34}$$

Utilizing Eqs. (4.31) and (4.34) into Eqs. (4.24) and (4.26), we get

$$(1+\frac{4}{3}R)\, A_3\, \xi\, \frac{d^2\theta}{d\xi^2} + ((1+\frac{4}{3}R)\, A_3 - Pr + \alpha\xi)\frac{d\theta}{d\xi} = 0 \tag{4.35}$$

The boundary conditions will be

$$\theta(0) = 0, \quad \frac{Pr}{\beta}\;\theta'(\frac{Pr}{\beta^2}) = Bi[1 - \theta(\frac{Pr}{\beta^2})] \tag{4.36}$$

Let us consider, $\frac{d\theta}{d\xi} = \psi$ and using in Eq. (4.35), we obtain

$$\frac{d\psi}{d\xi} + (\frac{(3+4R)\ A_3 - 3Pr}{(3+4R)A_3\xi} + \frac{3\alpha}{(3+4R)\ A_3})\psi = 0 \tag{4.37}$$

$$\Longrightarrow\ \frac{d}{d\xi}(\xi^{\frac{(3+4R)A_3 - 3Pr}{(3+4R)A_3\xi}} e^{\frac{3\alpha\xi}{(3+4R)A_3}}\psi) = 0 \tag{4.38}$$

After the integration of Eq. (4.38), substitution of ψ results the exact solution in the incomplete gamma function, that is,

$$\theta(\eta) = \frac{Bi\ (\Gamma(\frac{3Pr}{(3+4R)A_3},0)\ -\ \Gamma(\frac{3Pr}{(3+4R)A_3},\frac{3\alpha Pre^{-\beta\eta}}{(3+4R)A_3\beta^2}))}{\beta\ (\frac{3\alpha Pr}{(3+4R)A_3\beta^2})^{\frac{3Pr}{(3+4R)A_3}}\ e^{-\frac{3Pr\alpha}{(3+4R)a_3\beta^2}} + Bi\Gamma(\frac{3Pr}{(3+4R)A_3},0) - Bi\Gamma(\frac{3Pr}{(3+4R)A_3},\frac{3\alpha Pr}{(3+4R)A_3\beta^2})} \tag{4.39}$$

4.3 Results and discussion

In order to see the variations of nanofluids on physical parameters in the problem. It is observed that particle volume fraction is taken as 0.1 for the whole calculations. Figs. (22) - (25) depict variations of the S, Ξ, α and Φ on solution domain for β and change is seen in both the branches of solution. Varying the values of the parameters (S, Ξ, α *and* Φ) may affect the solution β accordingly. Increasing these parameters increase the upper branch else it decreases the lower branch. The arrows in the figures indicate the increase/decrease in upper and lower branch with the variable values of the parameters.

Figs. (26) - (29) indicate the change in skin friction with some parameters. The figures clearly show the increase in α causes a decrease in C_f whereas it has been seen that the increase in Ξ and Φ make an increase in C_f on both branches of solution. It is also illustrated that larger values of C_f is valid only for S.

In Figs. (30) - (33), the change in the velocity profile is being discussed that the effect of physical parameters are opposite on the upper and lower branch. Briefly, it can be described as the increase in S is an increase in lower branch and vice-versa. Similar behavior is seen in variations of α. On examining the effects of Ξ on the non-dimensional velocity distribution depict that increasing in Ξ is the increase in the resistance of flow in upper branch and vice-versa whereas no clear effect can be seen by varying the porous medium parameter, Φ.

Fig. (34) exhibits the change in local Nusselt number with suction parameter where the increase in α makes an increase in the Nu on upper and lower solutions. Figs. (35) and (36) illustrate the effects of non-dimensional temperature by varying the suction parameter and Biot number which clearly shows that the increase in S and Bi cause a decrease in dimensionless temperature profile due to thermal boundary layers. As a result, it seems obvious that the convective boundary layer gets decreased on both the branches. It can also been seen that the non-dimensional temperature profile depends upon the convective heat transfer coefficient due to its dependence and accumulated that the excess of heat transfer is also the excess of its coefficient on the profile.

Figs. (37) and (38) depict the effects of non-dimensional temperature due to Ξ and Φ on both branches in which Ξ is increased by dimensionless temperature for fixed value of suction parameter, $S = 1.5$, which has caused the increase in thermal boundary layer thickness. Whereas, Fig. (35), evidently shows that the change in Φ reduces the non-dimensional temperature in the same layer for both branches (upper and lower).

Table 1: Thermophysical properties of fluid and nanofluid

Physical properties	**Base fluid (water)**	**Nanofluid (Cu)**
C_p(J/kgK)	4179	385
ρ(kg/m^3)	997.1	8933
K(W/mK)	0.613	400

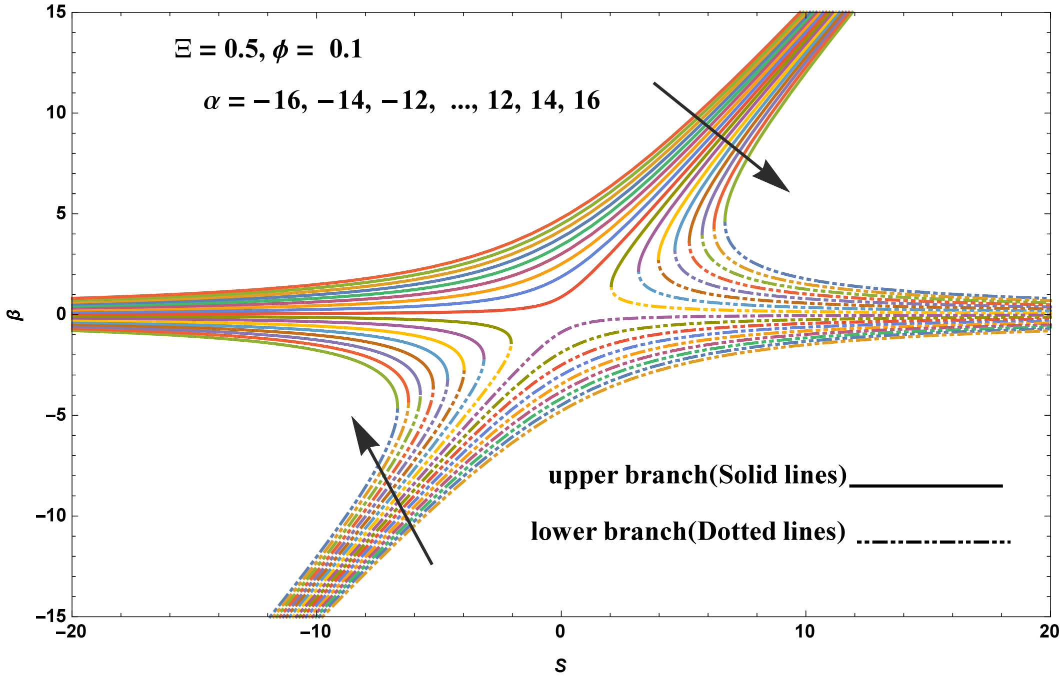


Fig. 22: Solution β as function of S

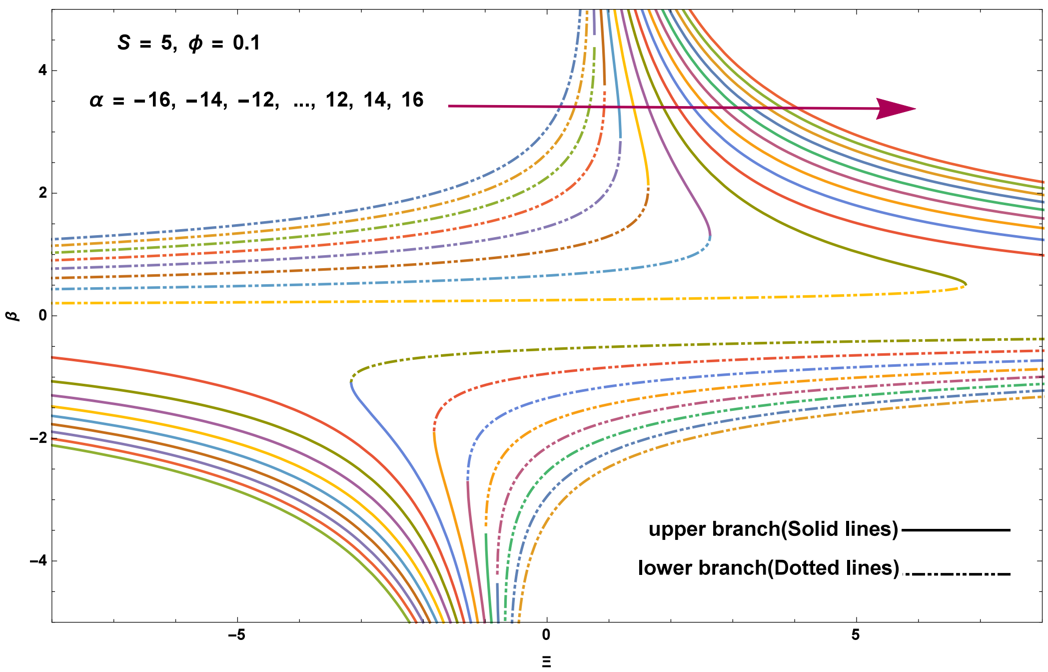


Fig. 23: Solution β as function of Ξ

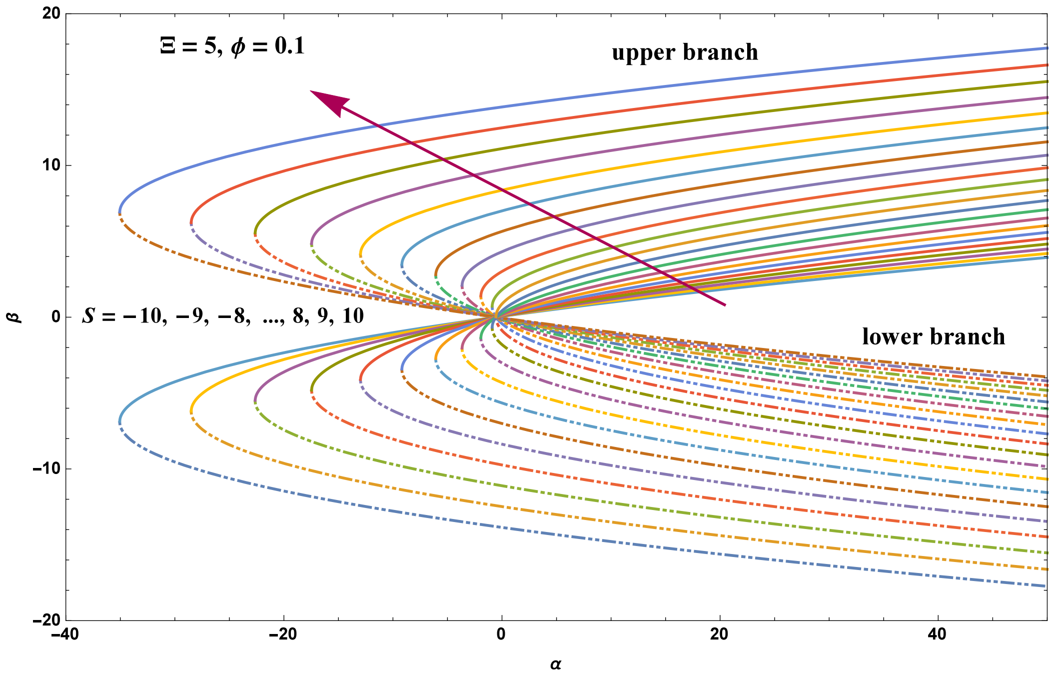


Fig. 24: Solution β as function of α

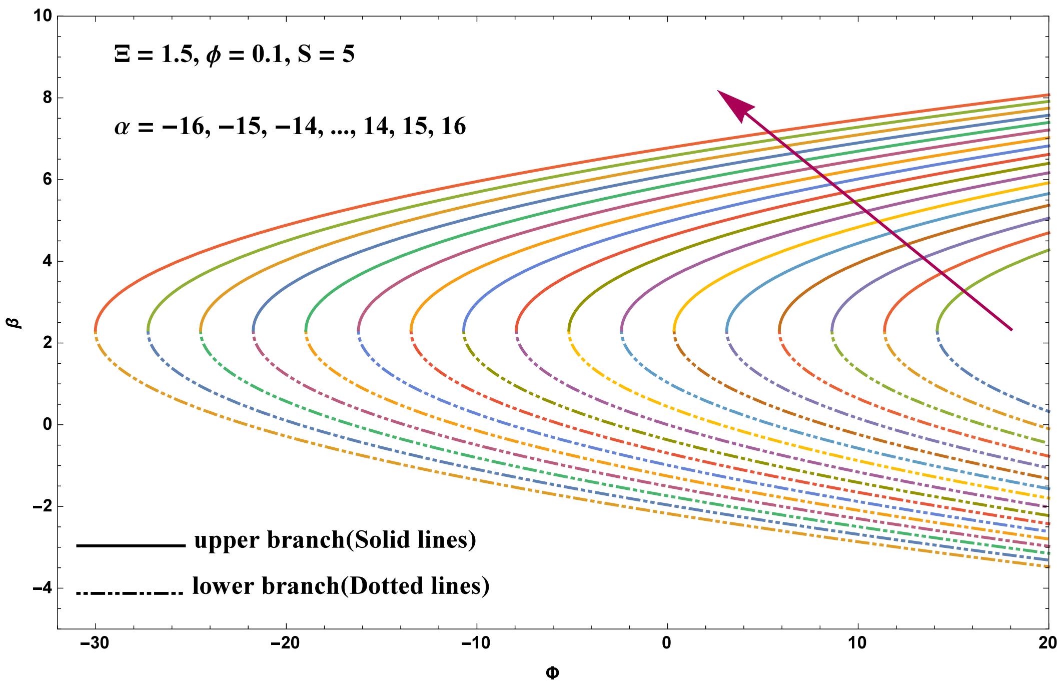


Fig. 25: Solution β as function of Φ

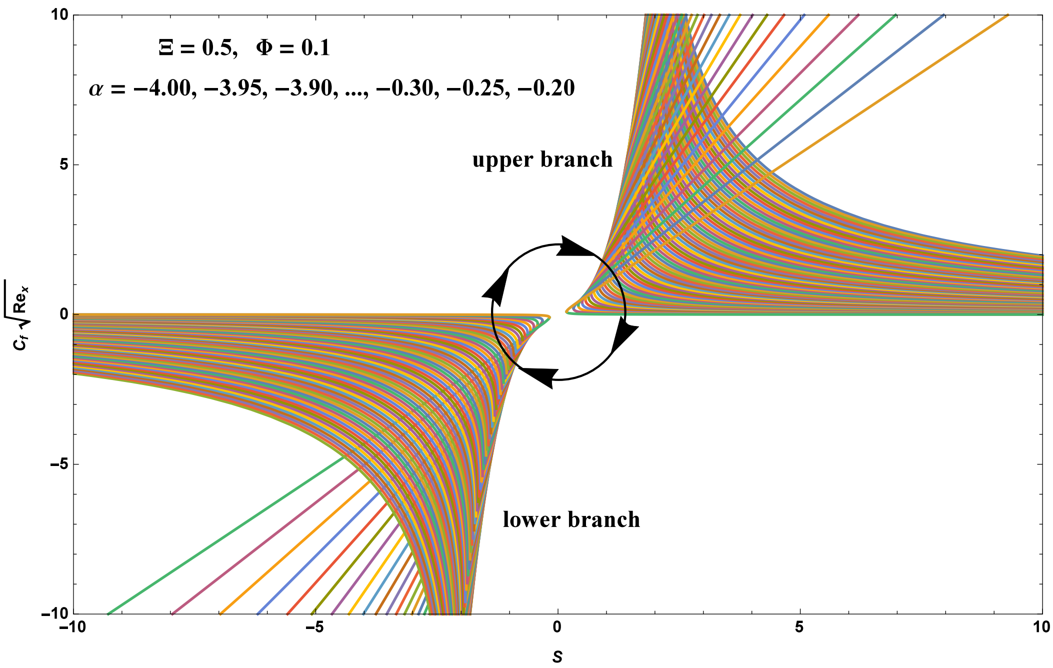


Fig. 26: Variations of skin friction with S

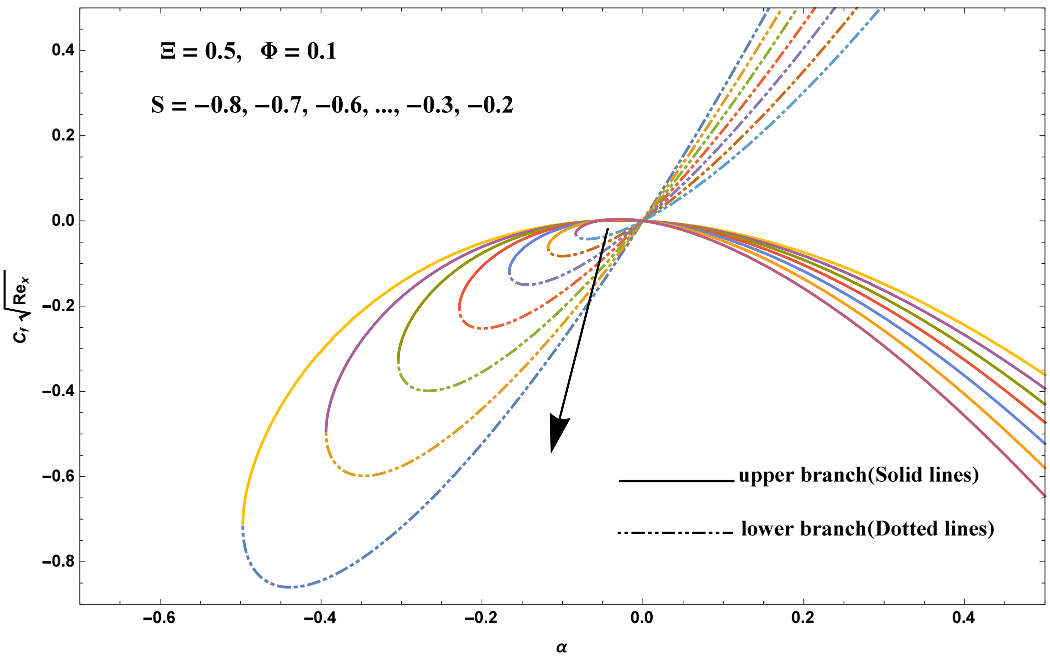


Fig. 27: Variations of skin friction with α

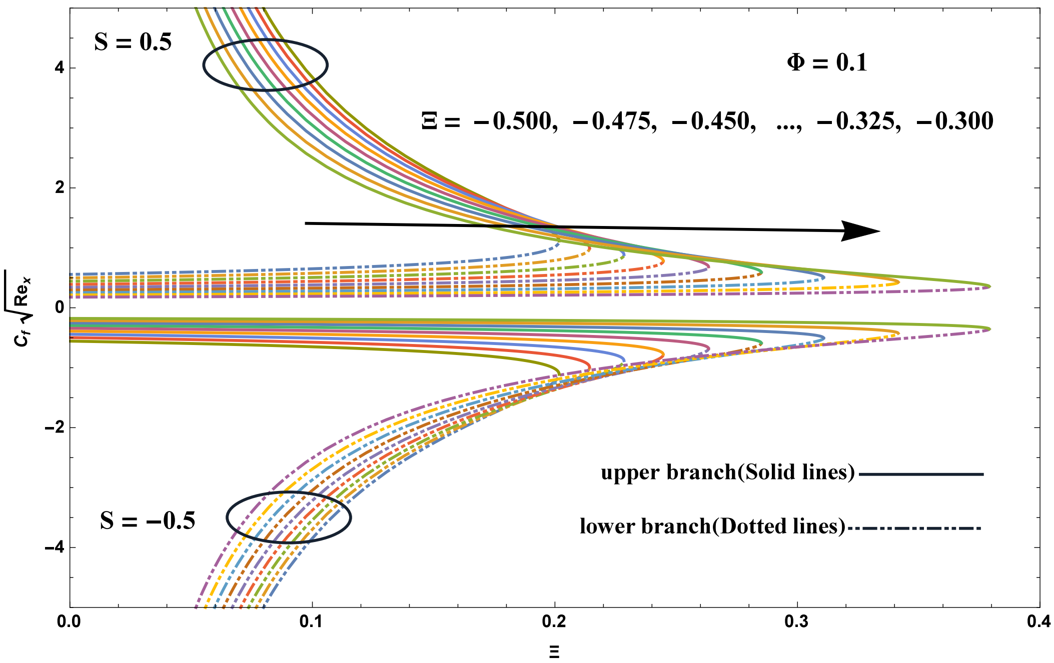


Fig. 28: Variations of skin friction with Ξ

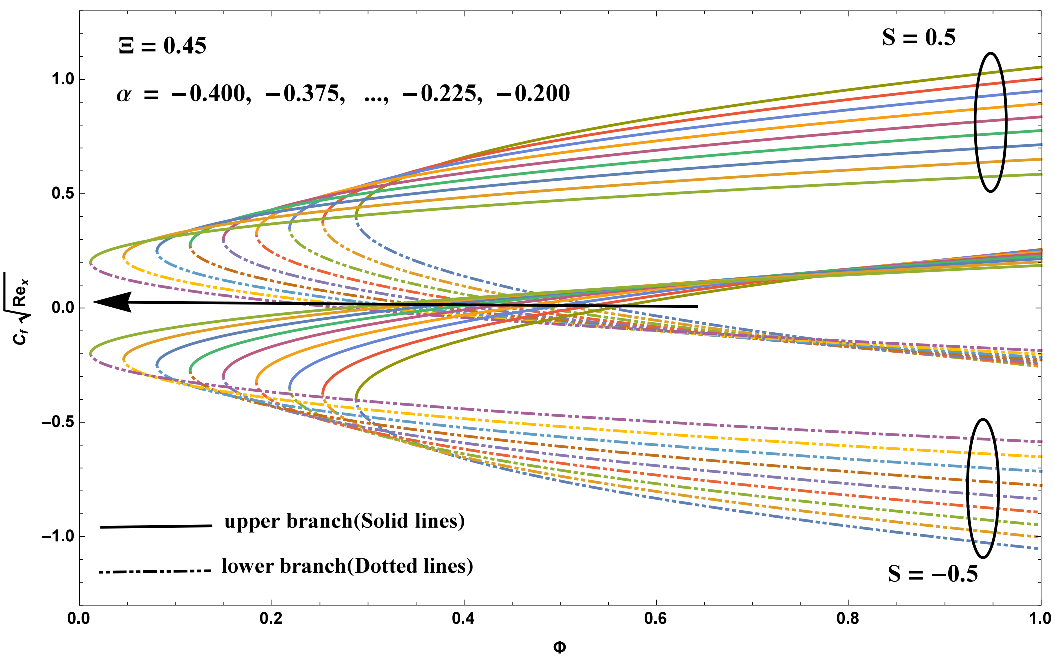


Fig. 29: Variations of skin friction with Φ

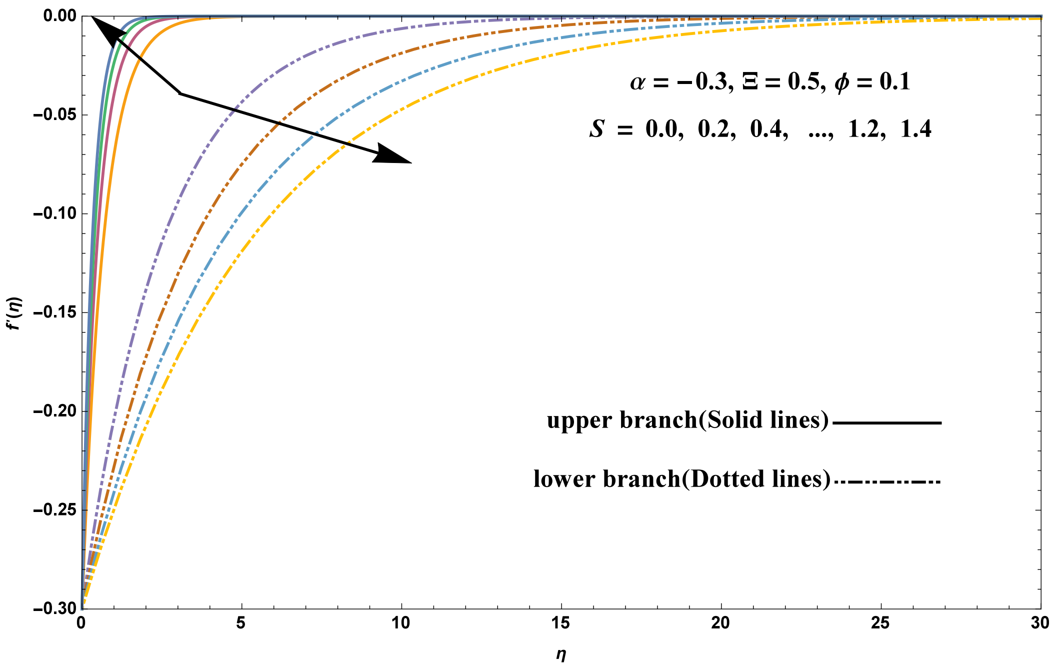


Fig. 30: Effects of suction parameter, S, on velocity profile

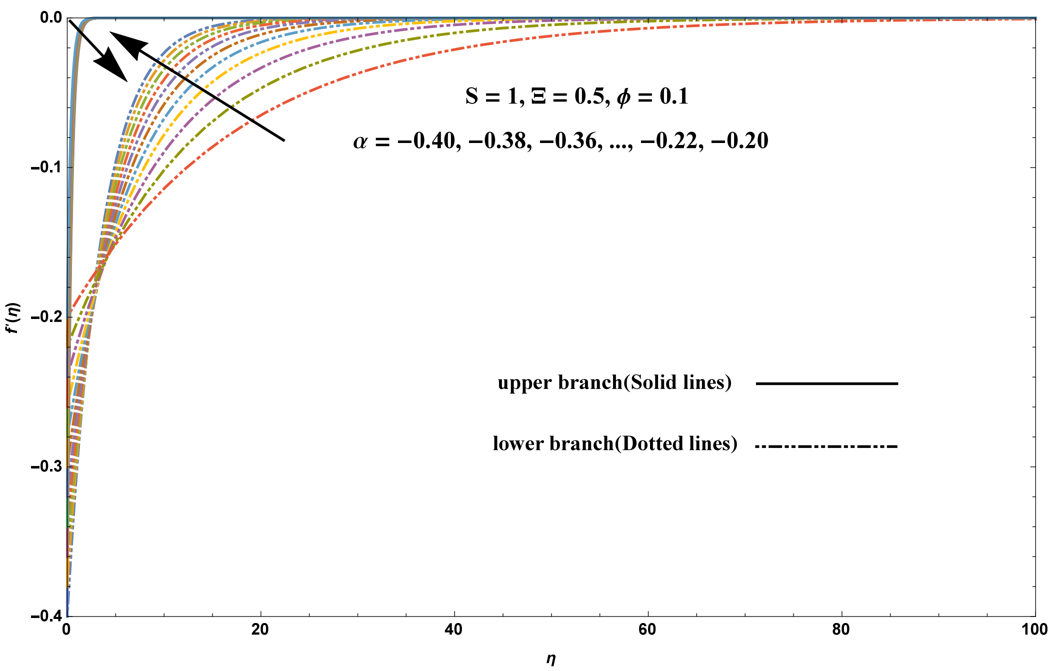


Fig. 31: Effects of stretching parameter, α, on velocity profile

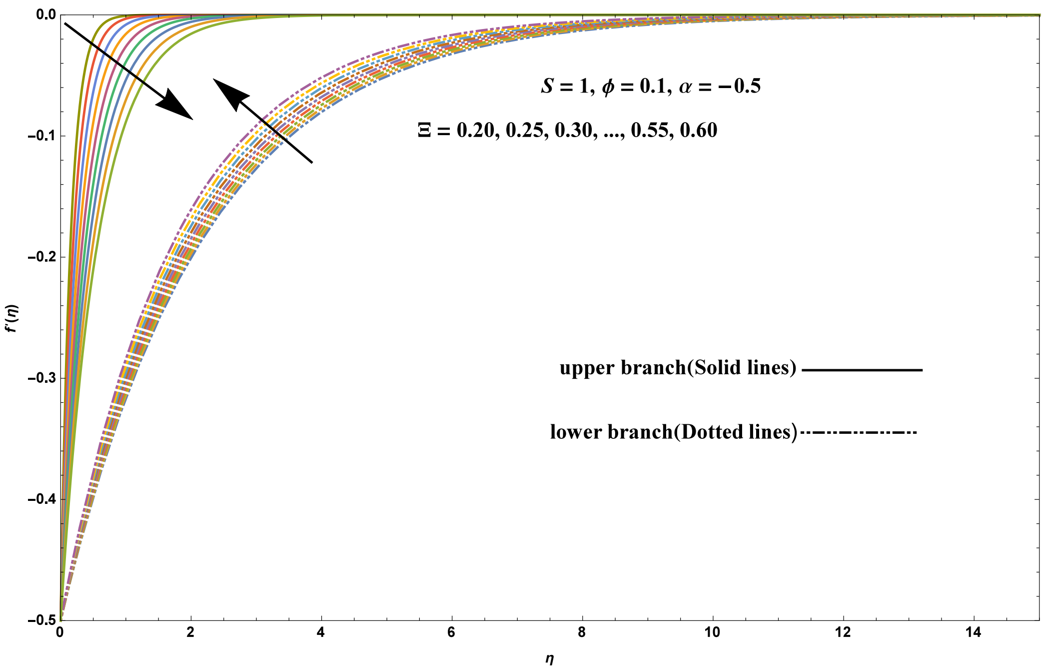


Fig. 32: Effects of viscosity ratio parameter, Ξ, on velocity profile

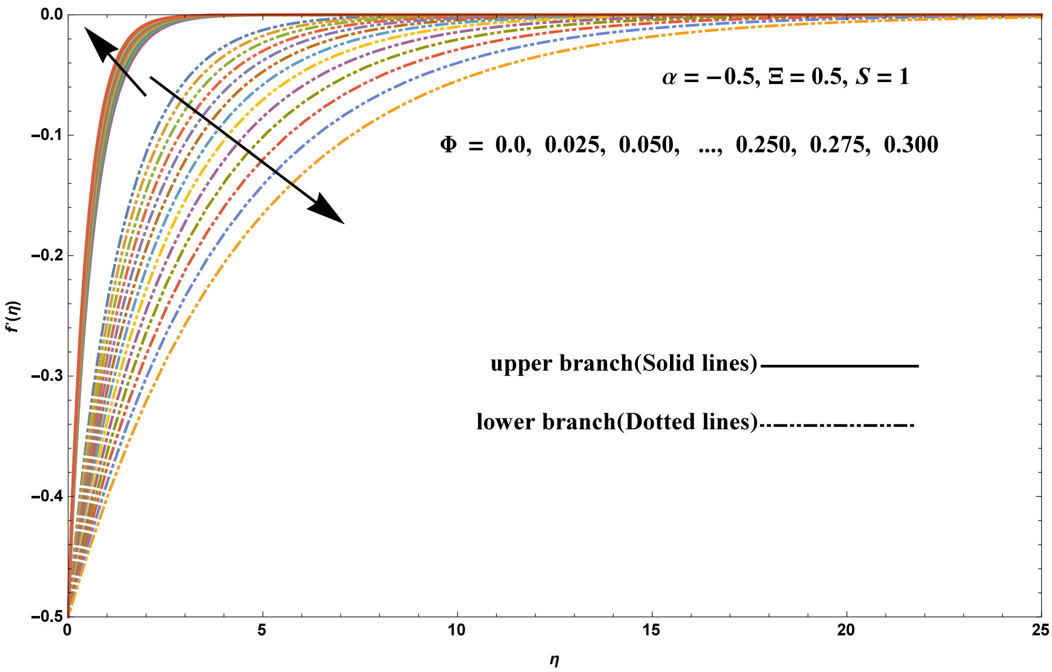


Fig. 33: Effects of porous medium parameter, Φ, on velocity profile

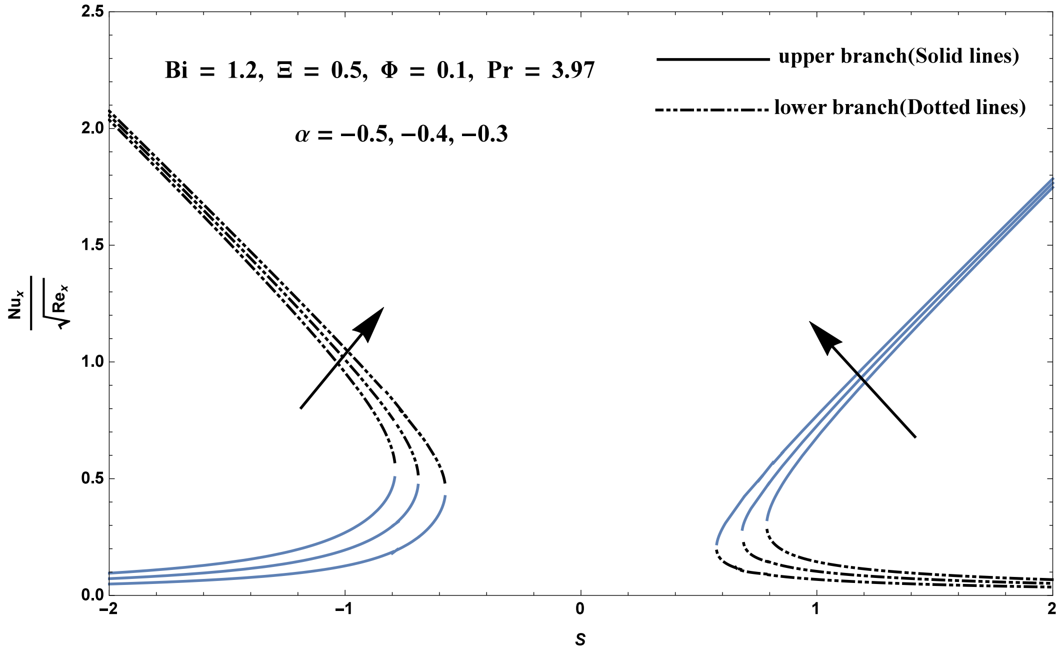


Fig. 34: Variations of Nusselt number with suction parameter, S

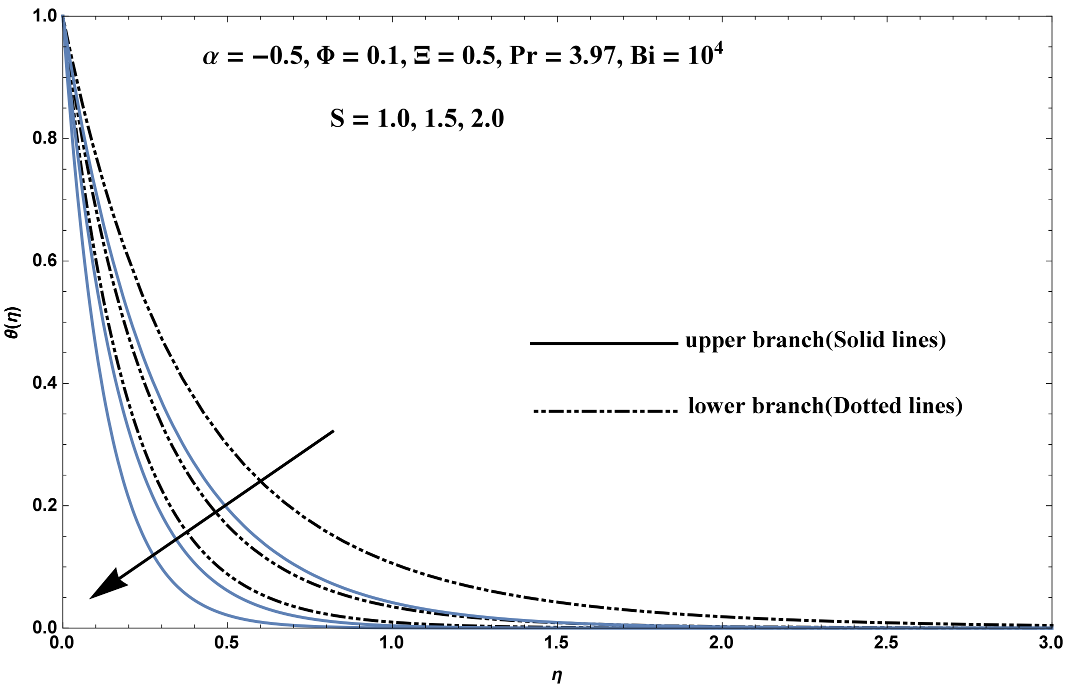


Fig. 35: Effects of suction parameter, S, on temperature profile

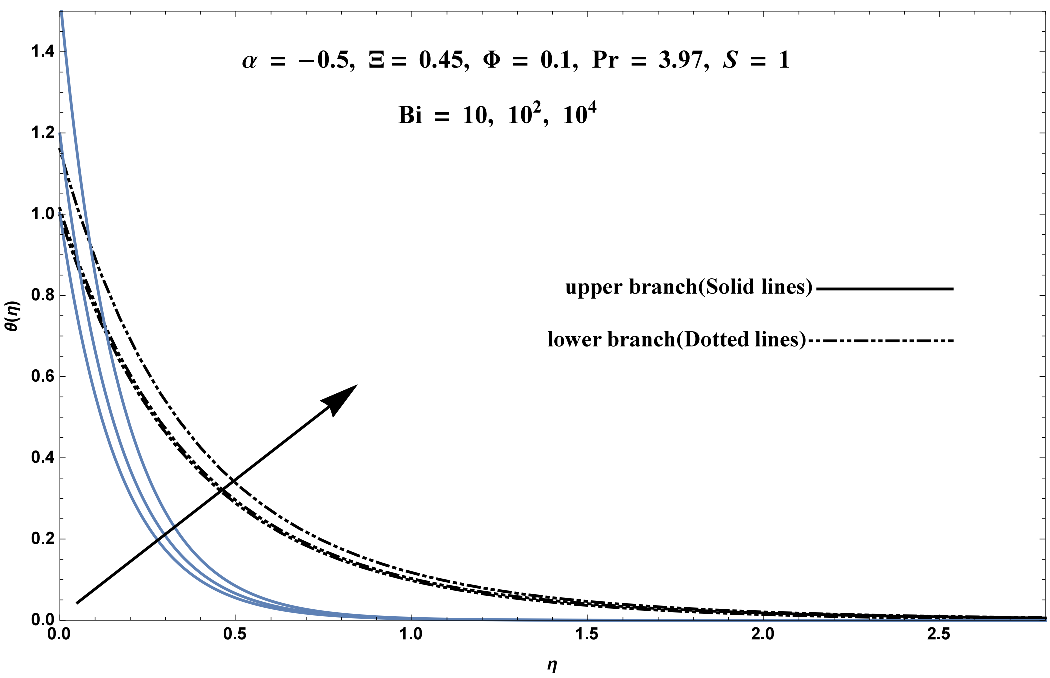


Fig. 36: Effects of Biot number, Bi, on temperature profile

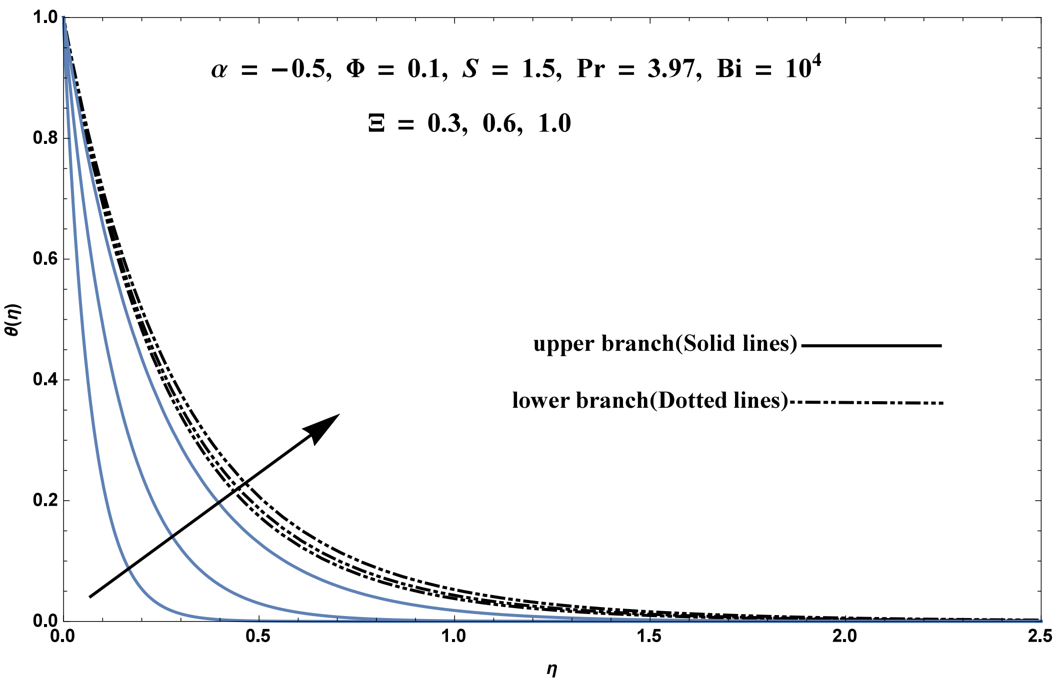


Fig. 37: Effects of viscosity ratio parameter, Ξ, on temperature profile

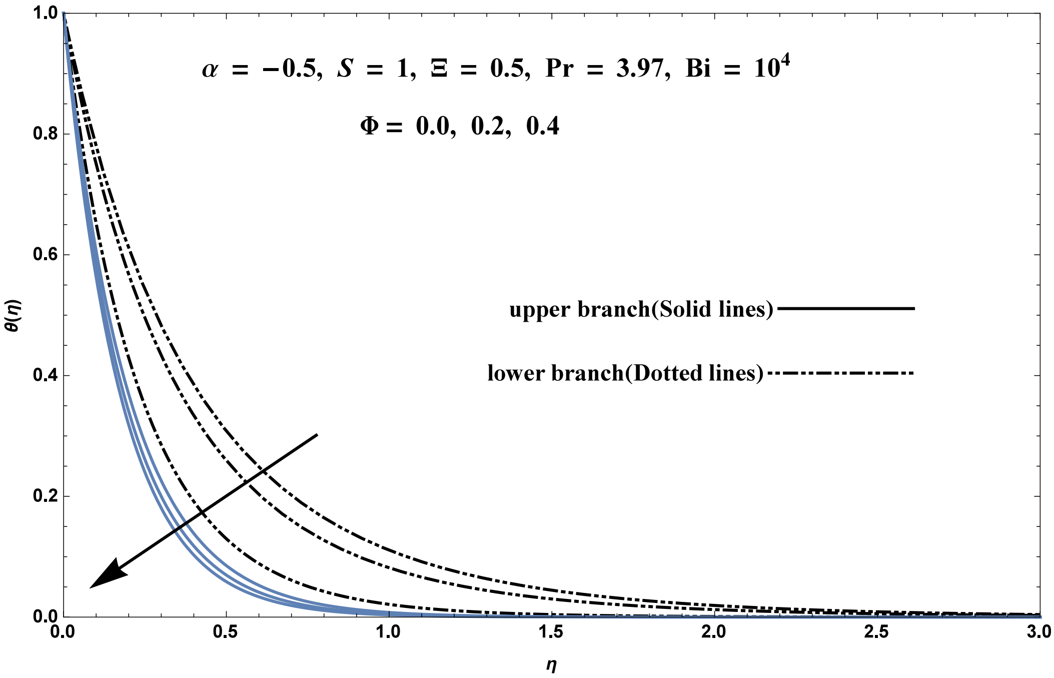


Fig. 38: Effects of porous medium parameter, Φ, on temperature profile

Chapter 5

Conclusion

This chapter is the conclusion of analytic and graphical results obtained from the review and extension work. All the findings from the above two chapters are discussed here in this chapter. The effect of parameters on velocity and temperature profiles including skin friction and Nusselt number are deduced in which the rise and fall, increase and decrease in the graphs are seen. Similar behavior is noticed in extension with an addition of nano-particle (Cu) with base fluid and thermal radiation, results are slightly different with the application of these additions. Following points are noted:

- For an exact solution, the governing PDEs are first converted into the nonlinear dimensionless ODEs through similarity transformation.
- Some assumptions yielded as dual nature solution in closed form from the dimensionless non-linear ODEs of momentum and energy.
- Basically, the effects of stretching α, viscosity ratio Ξ, mass suction S, porous medium Φ and Prandtl number Pr are described and shown in the figures for velocity, skin friction, streamlines, Nusselt number and energy profiles.
- Arrows in figures indicate the increasing or decreasing behavior of profiles with variations to the parameters applied.
- Cu-water nanofluid was considered to see some variations in the dual nature solution.
- Increase in the heat transfer occurred for volume fraction of nanoparticle [9], $\phi = 0, 0.1, 0.2$.

- The change in C_f is also identified by varying volume fraction of nanoparticle ϕ from 0 to 0.2.
- Increasing the value of ϕ also increased thermal boundary layer thickness.
- Amazing and quite interesting changes are seen in the profiles by varying the values of parameters.

References

[1] Sakiadis BC. Boundary-layer behavior on continuous solid surface: I. Boundary-layer equations for two-dimensional and axisymmetric flow. J AIChe, 7 (1961), 26 – 28.

[2] Sakiadis BC. Boundary-layer behavior on continuous solid surface: II. Boundary-layer equations for two-dimensional and axisymmetric flow. J AIChe, 7 (1961), 221 – 225.

[3] L.J. Crane, Flow past a stretching plate, Z. Angew. Math. Mech., 21 (1970), 645 - 647.

[4] M. A. Hossain, M. A. Alim, and D. A. S. Rees, "The effect of radiation on free convection from a porous vertical plate," Int. J. of Heat and Mass Transfer, 42 (1999), 181 – 191.

[5] V. Kumaran, G. Ramanaiah, A note on the flow over a stretching sheet, Acta Mech., 116 (1996), 229 - 233.

[6] I. Pop, D.B. Ingham, Convective Heat Transfer: Mathematical and Computational Modelling of Viscous Fluids and Porous Media, Oxford, Pergamon, 2001.

[7] D.A. Nield, A. Bejan, Convection in Porous Media, 3rd ed. Springer, New York, 2006.

[8] M. Miklavcic, C.Y. Wang, Viscous flow due to shrinking sheet, Quart. Appl. Math., 64 (June 2006), 283 - 290.

[9] Oztop HF, Abu Nada E. Numerical study of natural convection in potentially heated rectangular enclosures filled with nanofluids, Int. J. Heat fluid flow, 29 (2008), 1326 - 1336.

[10] Hayat T, Sajid M and Javed T, Nonlinear Dynamics, 51 (2008), 259 – 265.

[11] Muhaimin, Kandasamy R and Khamis A B, Applied Mathematics and Mechanics English Edition, 29 (2008), 1309.

[12] Wang C Y, International Journal of Non-Linear Mechanics, 43 (2008), 377.

[13] T. Fang, J. Zhang, Closed-form exact solutions of MHD viscous flow over a shrinking sheet, Commun. Nonlinear Sci. Num. Simul. 14 (2009), 2853 – 2857.

[14] A. Aziz, A similarity solution for laminar thermal boundary layer over a flat plate with a convective surface boundary condition, Comm. Nonlinear. Sci. Num. Simul. 14 (2009), 1064 – 1068.

[15] T. Fang, J. Zhang, S. yao, A new family of unsteady boundary layers over a stretching surface, Appl. Math. Comput. 217 (2010), 3747 - 3755.

[16] M.Z. Salleh, R. Nazar, I. Pop, Boundary layer flow and heat transfer over a stretching sheet with Newtonian heating, J. Thaiwan inst. Chem. Eng. 41 (2010), 651 – 655.

[17] W.A. Khan, I. Pop, Boundary layer flow of a nanofluid past a stretching sheet, Int. J. Heat Mass Transfer 53 (2010), 2477 – 2483.

[18] E. Magyari, Comment on "A similarity solution for laminar thermal boundary layer over a flat plate with a convective surface boundary condition" by A. Aziz, Comm. Nonlinear Sci. Numer. Simul. 16 (2011), 599 – 601.

[19] S. Yao, T. Fang, Y. Zhong, Heat transfer of a generalized stretching/shrinking wall problem with convective boundary conditions, Comm. Nonlinear. Sci. Num. Simul. 16 (2011), 752 – 760.

[20] S. Munawar, A. Mehmood, A. Ali, Time-dependent flow and heat transfer over a stretching cylinder, Chinese J. Phys., 50 (2012), 828 - 848.

[21] W.A. Khan, I. Pop, Heat Transfer Near Stretching Surface in Porous Medium Using Thermal Non-Equilibrium Model, AIAA J. Thermo-phys. Heat Transfer 26 (2012), 681 – 685.

[22] W.A. Khan, I. Pop, Boundary-Layer Stagnation-Point Flow Toward a Stretching Surface in a Porous Nanofluid-Filled Medium, J. Thermophys. Heat Transfer 26 (2012), 147 – 153.

[23] M. Qasim, Heat and mass transfer in a Jefferey fluid over a stretching sheet with heat source/sink, Alexandria Eng. J. 52 (2013), 571 – 575.

[24] H. A. Attia, A. L. Aboul-Hassan, M. A. M. Abdeen, A. El-Din Abdin, W. Abd El-Meged, Unsteady couette flow of a thermally conducting viscoelastic fluid under constant pressure gradient in a porous medium, Chinese J. Phys., 52 (2014), 1015 - 1027.

[25] Rizwan Ul Haq, D. Rajotia, and N.F.M. Noor, Thermophysical effects of water driven copper nanoparticles on MHD axisymmetric permeable shrinking sheet: Dual-nature study, Eur. Phys. J. E., 39 (2016).

[26] Y. Shagaiya Daniel, Z. A. Aziz, Z. Ismail, F. Salah, Effects of thermal radiation, viscous and Joule heating on electrical MHD nanofluid with double stratification, Chinese J. Phys, 55 (2017), 630-651.

[27] S. Saleem, M. Awais, S. Nadeem, N. Sandeep, T. Mustafa, Theoretical analysis of upper-convected Maxwell fluid flow with Cattaneo-Christov heat flux model, Chinese J. Phys, (2017).

[28] Fiaz Ur Rehman, S. Nadeem, Rizwan Ul Haq, Heat transfer analysis for threedimensional stagnation-point flow over an exponentially stretching surface, Chinese J. Phys, (2017).

[29] Feroz Ahmed Soomro, Rizwan-ul Haq, Zafar Hayat Khan, Qiang Zhang, Passive control of nanoparticle due to convective heat transfer of Prandtl fluid model at the stretching surface, Chinese J. Phys, (2017).

[30] Z.H.Khan,M. Qasim, Rizwan ul Haq, Qasem M.Al-Mdallal, Closed form dual nature solutions of fluid flow and heat transfer over a stretching/shrinking sheet in a porous medium, Chinese Journal of Physics, 55 (2017), 1284 - 1293.

Bahria University
Discovering Knowledge

MS-14A

Author's Declaration

I, ________ALI RAZA__________ hereby state that my MS thesis titled "_Dual nature study of convective heat transfers of nanofluid along a shrinking surface in a porous medium with thermal radiation.________________________________"

is my own work and has not been submitted previously by me for taking any degree from this university

________BAHRIA UNIVERSITY, ISLAMABAD________

or anywhere else in the country/world.

At any time if my statement is found to be incorrect even after my Graduate the university has the right to withdraw/cancel my MS degree.

Name of scholar: ________Ali Raza________
Date: ________19/06/2019________

Bahria University
Discovering Knowledge

MS-14B

Plagiarism Undertaking

I, solemnly declare that research work presented in the thesis titled "_ **Dual nature study of convective heat transfer of nanofluid along a shrinking surface in a porous medium with thermal radiation** ________" is solely my research work with no significant contribution from any other person. Small contribution / help wherever taken has been duly acknowledged and that complete thesis has been written by me.

I understand the zero tolerance policy of the HEC and Bahria University towards plagiarism. Therefore I as an Author of the above titled thesis declare that no portion of my thesis has been plagiarized and any material used as reference is properly referred / cited.

I undertake that if I am found guilty of any formal plagiarism in the above titled thesis even after award of PhD degree, the university reserves the right to withdraw / revoke my PhD degree and that HEC and the University has the right to publish my name on the HEC / University website on which names of students are placed who submitted plagiarized thesis.

Student / Author's Sign: ____________

Name of the Student: ALI RAZA

thesis

ORIGINALITY REPORT

17%	7%	11%	14%
SIMILARITY INDEX	INTERNET SOURCES	PUBLICATIONS	STUDENT PAPERS

PRIMARY SOURCES

1	Submitted to Higher Education Commission Pakistan Student Paper	5%
2	Z.H. Khan, M. Qasim, Rizwan Ul Haq, Qasem M. Al-Mdallal. "Closed form dual nature solutions of fluid flow and heat transfer over a stretching/shrinking sheet in a porous medium", Chinese Journal of Physics, 2017 Publication	1%
3	Submitted to Universiti Putra Malaysia Student Paper	1%
4	Submitted to National University of Singapore Student Paper	<1%